JOURNAL OF ALGEBRA 59, 77–171 (1979)

Groups which Admit a Fixed-Point-Free Automorphism of Order p^2

BENJAMIN RICKMAN

48 Bacon Lane, Kingsbury, London NW9, England

Communicated by I. N. Herstein

Received August 10, 1977

1. INTRODUCTION

We give a proof of the following theorem.

MAIN THEOREM. *Let G be a finite group and p be a prime. Assume that G admits a fixed-point-free automorphism α of order p^2. Then G is solvable.*

An important new tool that is brought to attack this problem is G. Glauberman's classification of finite groups that do not involve the symmetric group of degree 4 [7]. The methods of Higman and Stewart [16] and the classification of finite simple groups with no elements of order 6 [25, 6, 7], are also important. Yet further tools are the results of Section 3 which closely resemble, and were inspired by G. Glauberman [8] and M. J. Collins [2, Theorem 9.3].

In order to greatly simplify the original proof obtained of this result, we use the following result of G. Glauberman which has only recently been obtained [26].

THEOREM 1.1. *Let G be a finite group and p be an odd prime. Let P be a Sylow p-subgroup of G. Then*

$$P \cap G' = \langle P \cap N(K)' \colon 1 \neq K \text{ char. } P \rangle.$$

The precise set of characteristic subgroups of P required to make Theorem 1.1 work is complicated to define. We shall not need any more refined statement.

Several authors have obtained the conclusion of the main theorem under additional hypotheses. Gorenstein and Herstein [12] did the case $p = 2$. Pettet [21] and Carr [1] independently settled the case $C_G(\alpha^p)$ has odd order.

The main theorem is a contribution to the continuing program of showing that finite groups which admit a fixed-point-free group of automorphism are solvable.

0021-8693/79/070077–95\$02.00/0

We now discuss the proof of the main theorem. We assume that the main theorem is false and take a counterexample G to the main theorem of least order. In Section 6 we show that G cannot exist.

To fix ideas, set $\beta = \alpha^p$ and $F = C_G(\beta)$. We observe that $p \nmid |G|$, so that Theorem 6.2.2 of [10] applies to G. It follows that G has one and only one α-invariant Sylow q-subgroup of G for each prime q that divides $|G|$. As remarked above, the case $p = 2$ is settled in [11]. So we can suppose that p is odd. By minimality of G and Proposition 3.1 of [1], every α-invariant proper subgroup H of G can be expressed as $F(H) \cdot C_H(\beta)$.

Now α acts on F as a fixed-point-free automorphism of prime order. By a theorem of Thompson [10, Theorem 10.2.1], F is nilpotent. So G can also be considered as a finite group which admits an automorphism (β in this case) of prime order, p, whose fixed-point-set is a nilpotent p'-group.

It has been conjectured that all such groups are solvable [2, p. 135]. This problem has been considered by several authors including Collins [2, and 3]. Results of this type make up Section 5. The results of this section contain the main results required for the analysis in Section 6.

In analogy with Sect. 3 of [3], we require some non-simplicity criteria. These are results 4.2, 4.3 and 4.4 stated at the beginning of Section 4. This section consists of proofs of these results and statements of well-known results that are frequently used, and the proofs of 4.2, 4.3 and 4.4. The proofs of 4.2, 4.3 and 4.4 employ some of the results and techniques of Higman and others, as in [6, 16].

Let P denote the unique α-invariant Sylow 3-subgroup of G. We deduce from [7] that $P \neq 1$, and can conclude that $p \geqslant 5$. Let $W^* = \Omega_1(Z(P)) \cap F$ and define W as follows. If $W^* \neq 1$, let $W = W^*$. Otherwise let $W = \Omega_1(Z(P))$.

In Section 6 we apply Theorem 1.1 to prove that if q is an odd prime dividing $|G|$ and Q is the α-invariant Sylow q-subgroup of G, then $N(Q)$ is a maximal α-invariant subgroup of G. In particular $N(P)$ is a maximal α-invariant subgroup of G and we show that $N(W) = N(P)$.

Set $K = C_{N(P)}(\beta)$, $\pi_0 = \pi(N(P)/F(N(P)))$, $\pi = \pi_0 - \{2\}$ and $\pi^* = \{q \in \pi: O_q(N(P)) \neq 1\}$. We find, using 3.3, that if $O_\pi(K)$ is a Hall π-subgroup of $N(P)$, then W is weakly closed in P with respect to G. We also derive that if q is a prime divisor of $|F|$ and Q is the unique α-invariant Sylow q-subgroup of G, then $F \leqslant N(Q)$. Using Theorem 6.2.2 of [10], it follows that Q is also the only β-invariant Sylow q-subgroup of G. This useful and possibly well-known result is the basis used in the formulation of Hypothesis 5.1 in Section 5.

We also find that one of (1), (2) and (3) holds

(1) W is weakly closed in P with respect to G and $F \leqslant N(P)$.

(2) W is weakly closed in P with respect to G, $O_2(N(P)) = 1$ and F is a $3'$-group. Furthermore each W-invariant $3'$-subgroup of G is centralized by W.

(3) F is a $3'$-group and whenever H is a β-invariant solvable subgroup of G such that $P \cap H \in S_3(H)$ and $Z(K) \leqslant H$ then $P \cap H \trianglelefteq H$. Furthermore,

if W is not weakly closed in P with respect to G, then the following holds. If $q \in \pi^*$, Q is the unique α-invariant Sylow q-subgroup of G and $W_Q = \Omega_1(Z(Q)) \cap F$, then W_Q is weakly closed in Q with respect to G, $W_Q \leqslant Z(K)$ and $N(W_Q) = N(Q) = QF$.

In order to obtain the condition on W-invariant $3'$-subgroups of G mentioned in (2) we apply result 5.4 in Section 5. This result is proved by an argument similar to the proof of Theorem 3.1 of [13], and uses an observation of G. Glauberman [2, p. 130]. In the discussion leading to the condition on β-invariant solvable subgroups of G, mentioned in (3), we make use of results 2.11, 2.14 and 2.16 of Section 2.

The results of Section 2 are results on solvable groups that admit an automorphism with prescribed properties. These were inspired by unpublished work of G. Higman and N. K. Dickson.

The need to eliminate possibilities (1), (2) and (3) leads to the formulation of results 5.6, 5.7 and 5.5 respectively. A minimal counterexample to 5.5 or 5.6 satisfies Hypothesis 5.8 of Section 5. The main part of Section 5 is taken up by the proof of Proposition 5.14, which says that no group can satisfy Hypothesis 5.8. The proof of 5.14 uses results 4.2 and 4.3 of Section 4. Result 5.7 is almost immediate from 4.4. We have shortened Section 5 by using a recent and interesting theorem of McBride [19].

We should like to thank the many pepople in Chicago and Oxford with whom we have discussed this problem.

The notation is that of D. Gorenstein, "Finite Groups" [10], with the following additions. For a finite group X, subgroups $K, H \leqslant X$ such that $K \trianglelefteq H$ and a set of primes π, we let O_π (H mod K) be the pre-image in H of $O_\pi(H/K)$. For $H \leqslant X$ we let $\mathrm{Aut}_X(H)$ be the quotient group $N_X(H)/C_X(H)$. An $F[X]$-module is a vector space over the field F on which X acts as a (not necessarily faithful) group of linear transformations. For a prime p, we let $S_p(X)$ denote the set of Sylow p-subgroups of the group X. A p-nilpotent group is a group with a normal p-complement.

In this paper the term "group" should be taken to mean "finite group" and the term "simple group" should be taken to mean "finite non-abelian simple group."

2. Results on Solvable Groups

In this section, we obtain the necessary results on the structure of solvable groups that arise in the proof of the theorem. We will also obtain some general results which will be applied often in later sections. We begin by restating Proposition 3.1 of [1].

THEOREM 2.1. *Let S be a solvable group and p be an odd prime. Assume that S admits a fixed-point-free automorphism α of order p^2. Then $S = F(S)\, C_S(\alpha^p)$.*

The next results are well known and useful.

LEMMA 2.2. *Let G be a group and A be a group of operators on G such that $(|A|, |G|) = 1$. Let $H \leqslant C_G(A)$. Then $N_G(H) = C_G(H).(N_G(H) \cap C_G(A))$.*

Proof. Now $[H, N(H)] \leqslant H$ and $[A, H] = 1$. So $[H, N(H), A] = [A, H, N(H)] = 1$. By Theorem 2.2.3 of [10], $[N(H), A, H] = 1$. So, by Theorem 6.2.2 (iv) of [10], the lemma follows.

LEMMA 2.3. *Let p be a prime and G be a group. Assume that G admits an automorphism α of order p such that $C_G(\alpha)$ is a p'-group. Then the following holds*

(i) *G is a p'-group*

(ii) *Let A be an α-invariant normal subgroup of G and B be an α-invariant subgroup of G such that $G = AB$. Then $C_G(\alpha) = C_A(\alpha)\, C_B(\alpha)$.*

Proof. (i) is clear as the elements of $G - C_G(\alpha)$ are permuted by α in orbits of length p.

We now prove (ii). By Theorem 2.4.1 of [15], G/A and $B/(A \cap B)$ are α-isomorphic. So $C_{G/A}(\alpha) \cong C_{B/(A \cap B)}(\alpha)$. By (i) and Theorem 6.2.2 (iv) of [10],

$$C_{B/(A \cap B)}(\alpha) \cong C_B(\alpha)/(C_A(\alpha) \cap C_B(\alpha)) \cong C_A(\alpha)\, C_B(\alpha)/C_A(\alpha).$$

Also $C_{G/A}(\alpha) \cong C_G(\alpha)/C_A(\alpha)$. Comparing orders and using the relation $C_A(\alpha)\, C_B(\alpha) \leqslant C_G(\alpha)$, we have (ii). The result is proved.

We next state, without proof, some well-known results from representation theory.

LEMMA 2.4. *Let G be a group, F be a field and M be a faithful and irreducible $F[G]$-module. Let K be a finite extension field of F and $M^* = M \otimes_F K$. Let N be an irreducible $K[G]$-submodule of M^*. Then*

(i) *$M^* = \langle N^\sigma : \sigma \in \mathrm{Gal}(K:F) \rangle$.*

(ii) *N is a faithful $K[G]$-module.*

(iii) *Let $x \in G$. Then $C_{M^*}(x) = C_M(x) \otimes_F K$.*

(iv) *Let $H \leqslant G$. Then $C_{M^*}(H) = C_M(H) \otimes_F K$.*

The next result appears as Theorem 7.6 of [2] and appears as (part of) Lemma 2.8 of [3].

THEOREM 2.5. *Let p, q, r be three distinct primes. Let $H = X \cdot A$ where X is an extra-special q-group and $A = \langle \alpha \rangle$ is cyclic of order r. Assume that $C_X(\alpha) = X'$. Assume further that H acts faithfully and irreducibly on a vector space V over $GF(p)$ in such a way that $C_V(A) = 0$.*

Then $q = 2$ and there is an integer n such that $|X| = 2^{2n+1}$ and r is the Fermat prime $2^n + 1$.

We shall use the next result many times in what follows.

LEMMA 2.6. *Let p be a prime and G be a p'-group. Assume that G admits an automorphism α of order p such that $C_G(\alpha)$ is nilpotent. Assume that, for some prime q, $O_q(C_G(\alpha)) \in S_q(G)$. Then G is q-nilpotent.*

Proof. Let $X \leqslant O_q(C_G(\alpha))$. By 2.2, $N(X)/C(X)$ is a q-group. But now, by Theorem 7.4.5 of [10], G is q-nilpotent. The lemma is proved.

We now discuss solvable groups which satisfy the following hypothesis. We will work under this hypothesis up to the last lemma of Section 2.

HYPOTHESIS 2.7. S is a solvable group which admits an automorphism α of prime order, p. Assume that $F = C_S(\alpha)$ is a nilpotent p'-group. For any prime q, let $F_q = O_q(F)$.

By 2.3(i), $p \nmid |S|$. We require three technical results.

LEMMA 2.8. *Assume that $S = QUV$, where $Q = F(S)$ is an elementary abelian q-group, U is an α-invariant special r-group and V is a non-identity α-invariant t-group, for primes q, r, t such that $q \neq r \neq t$. Assume that*

(a) *$U \cdot V \cdot \langle\alpha\rangle$ acts faithfully and irreducibly on Q.*

(b) *$V \cdot \langle\alpha\rangle$ acts irreducibly on $U/\Phi(U)$, or $U = 1$.*

(c) *α acts irreducibly on $V/C_V(U)$, an elementary abelian group.*

(d) *$C_U(V) = \Phi(U)$.*

(e) *$C_Q(\alpha) = 1$.*

Then $U = 1$.

Proof. Suppose that $U \neq 1$. Clearly $U \in S_r(S)$ and so, by Theorem 6.2.2 of [10], $F_r \leqslant U$. By Theorem 5.2.3 of [10] and 2.3(ii), $(Q \cdot [Z(U), \alpha]) \cap F = 1$. By Theorem 10.2.1 of [10] and (a), $[Z(U), \alpha] = 1$. As U is special, $U' \leqslant F_r$, so that $F_r \trianglelefteq U$. By 2.2,

$$U = F_r \cdot C_U(F_r). \tag{1}$$

If $U = F_r$ then, as $r \neq t$ and F is nilpotent, by 2.2, $U = C_U(V)$. By (d), $U = \Phi(U)$ and so $U = 1$, a contradiction. So $U \gneqq F_r$. We note too that if U is abelian, $U = Z(U) = F_r$. So U is not abelian. So, by (d), $Z(U) \leqslant Z(U \cdot V \cdot \langle\alpha\rangle)$. By (a) and Theorem 3.2.2 of [10], $Z(U)$ is cyclic. So

$$U \text{ is extra-special.} \tag{2}$$

As $Z(U) = U' \leqslant Z(F_r)$, $Z(F_r) \trianglelefteq U$. By 2.2, $Z(F_r) = Z(U)$. By (1), $Z(C_U(F_r)) = Z(U)$. As $U \gneqq F_r$, by (6), $C_U(F_r) \nleqslant F_r$. As $Z(U) \leqslant F_r$, $C_U(F_r)$ is not abelian. So, using (2), we see that

$$C_U(F_r) \text{ is extra-special and } Z(C_U(F_r)) = Z(U). \tag{3}$$

By (a), as $Z(U) \leqslant Z(U \cdot V \cdot \langle\alpha\rangle)$, $C_Q(Z(U)) = 1$. Let Q_0 be an irreducible $C_U(F_r)\langle\alpha\rangle$-subgroup of Q. Then, by (e) and (3), $C_U(F_r)\langle\alpha\rangle$ acts faithfully on Q_0. So, by 2.5, $r = 2$ and there is an integer n such that $|C_U(F_r)| = 2^{2n+1}$ and $p = 2^n + 1$.

If $F_r \gneqq Z(F_r)$ then, using (2), as $Z(U) = Z(F_r)$, F_r is extra-special. If $F_r = Z(F_r)$, as $Z(U) = Z(F_r)$, $F_r \simeq Z_r$. So, as $r = 2$, there is a non-negative integer m such that

$$|F_r| = 2^{2m+1}. \tag{4}$$

Suppose now that $C_{V/C_V(U)}(\alpha) \neq 1$. By (c) and Theorem 6.2.2(iv) of [10], $V = C_V(U) \cdot C_V(\alpha)$. Let $X = C_V(\alpha)$. By (d), $C_U(X) = \Phi(U)$. As $r \neq t$ and F is nilpotent, $F_r = Z(U)$. So $U = C_U(F_r)$. By Corollary 33.6(3) of [4], U has $2^{2n} \pm 2^n$ elements of order 4. As α fixes no element of order 4 and $p = 2^n + 1$, U has $2^n p$ elements of order 4. As X fixes no element of order 4, $t \mid 2^n p$. But $t \neq r$ and $t \neq p$, a contradiction.

So $C_{V/C_V(U)}(\alpha) = 1$ and $(V/C_V(U)) \cdot \langle\alpha\rangle$ is a Frobenius group. Let $M = U/\Phi(U)$ and K be a (finite) splitting field for $(V/C_V(U)) \cdot \langle\alpha\rangle$ which contains $GF(r)$. Let $Z = GF(r)$ and $M^* = M \otimes_Z K$. By (b), $(V/C_V(U)) \cdot \langle\alpha\rangle$ acts faithfully and irreducibly on M. So, by 2.4, M^* is completely reducible and $(V/C_V(U)) \cdot \langle\alpha\rangle$ acts faithfully on each irreducible submodule of M^*. Let N be an irreducible $(V/C_V(U)) \cdot \langle\alpha\rangle$-submodule of M^*. By Theorem 3.4.3 of [10], there is a homogeneous component, W, of N with respect to $V/C_V(U)$ such that

$$N = W \oplus W\alpha \oplus \cdots \oplus W\alpha^{p-1},$$

and

$$C_N(\alpha) = W(1 + \alpha + \cdots + \alpha^{p-1}).$$

We deduce that $\dim_K N = p \dim_K C_N(\alpha)$. As M^* is completely reducible, by 2.4, (4) and Theorem 6.2.2 of [10], $\dim_Z M = \dim_K M^* = p \dim_K C_{M^*}(\alpha) = p \dim_Z C_M(\alpha) = 2mp$.

So $|U| = 2^{2mp+1}$. Now, by (1), (2), (3) and (4), as $|C_U(F_r)| = 2^{2n+1}$ $|U| = 2^{2(m+n)+1}$. So $mp = m + n$. As $p = 2^n + 1$, we get

$$2^n = n/m \leqslant n,$$

(note $m > 0$), in contradiction with a celebrated theorem of Cantor. The lemma follows.

The next result is an argument due to G. Higman.

LEMMA 2.9. *Let r, t be distinct primes which are both different from p. Suppose that $S = UV$ where U is an α-invariant non-identity r-group and V is a non-identity α-invariant t-group. Assume further that $U \trianglelefteq S$ and that*

(a) $C_{Z(U)}(\alpha) = 1$.

(b) *$S\langle\alpha\rangle$ is faithfully and irreducibly represented on a vector space M over a splitting field K for $S\langle\alpha\rangle$.*

(c) *$C_S(\alpha)$ acts trivially on $C_M(\alpha)$.*

Then $C_S(\alpha) = 1$.

Proof. By Theorem 3.4.1 of [10], there are a or p Wedderburn components of M with respect to S.

Suppose that there is one Wedderburn component of M with respect to S. By Theorem 3.4.1 of [10] applied to $U \cdot V$ acting on an irreducible submodule of M, we see that the number of isomorphism classes of irreducible U-submodules of M is a power of t. So α fixes a Wedderburn component, W, of M with respect to U. Let K be the kernel of the action of $U \cdot \langle\alpha\rangle$ on W.

If $Z(U) \leqslant K$ then, by Theorem 3.4.1(iii) of [10], $Z(U)$ acts trivially on M and so, by (b), $Z(U) = 1$. But $U \neq 1$, a contradiction. So $Z(U) \nleqslant K$.

But, by Lemma 3.2.1 of [10], $Z(U)K/K$ is represented on W as a group of scalar matrices. So αK centralises $Z(U)K/K$. So, by (a), $Z(U) = [Z(U), \alpha] \leqslant K$, a contradiction. We conclude that there are p Wedderburn components of M with respect to S.

Let X be a Wedderburn component of M with respect to S. Then, by Theorem 3.4.1 of [10],

$$M = X \oplus X\alpha \oplus \cdots \oplus X\alpha^{p-1}. \tag{5}$$

We deduce that

$$C_M(\alpha) = X(1 + \cdots + \alpha^{p-1}). \tag{6}$$

Let $x \in X$ and $y \in C_S(\alpha)$. Then, by (c),

$$x(1 + \cdots + \alpha^{p-1}) = x(1 + \cdots + \alpha^{p-1})\, y = xy(1 + \cdots + \alpha^{p-1}).$$

As $xy \in X$, by (5), $xy = x$. We deduce that $C_S(\alpha)$ acts trivially on M. By (b), $C_S(\alpha) = 1$. The lemma is proved.

LEMMA 2.10. *Let σ be a set of primes. Assume that S does not have σ-length 1, but that every α-invariant proper subgroup and every quotient of S by a non-identity α-invariant normal subgroup of S has σ-length 1. Then there exists primes q, r, t such that $q, t \in \pi(S) \cap \sigma$ and $r \in \pi(S) \cap \sigma'$. There also exist non-identity α-invariant subgroups Q, U, V such that Q is a q-group, U is an r-group and V is a t-group, and the following conditions hold.*

(i) *$Q = F(S)$ and Q is elementary abelian.*

(ii) *$U \cdot V \cdot \langle\alpha\rangle$ acts faithfully and irreducibly on Q.*

(iii) *U is a special r-group.*

(iv) *$V\langle\alpha\rangle$ acts faithfully on $U/\Phi(U)$.*

(v) *$V/C_V(U)$ is an elementary abelian group acted on irreducibly by α.*

(vi) *$C_U(V) = \Phi(U)$ and $U = [U, V] \neq 1$.*

(vii) *If $q = t$ then $C_V(U) = 1$.*

(viii) *If $C_V(\alpha) \nleq C_V(U)$ then there is an $x \in F$ such that $V = \langle x\rangle$.*

(ix) *$C_Q(\alpha) \neq 1$.*

(x) *$C_{\Phi(U)}(\alpha) = 1$.*

Proof. If $O_{\sigma'}(S) \neq 1$ then $S/O_{\sigma'}(S)$ has σ-length 1. But then S has σ-length 1, a contradiction. So

$$O_{\sigma'}(S) = 1. \tag{7}$$

If $O_{\sigma,\sigma',\sigma}(S)$ has σ-length 1, then $O_{\sigma,\sigma',\sigma}(S) = O_{\sigma,\sigma'}(S)$. By Theorem 6.2.2 of [10] applied to $S/O_{\sigma,\sigma'}(S)$, $S = O_{\sigma,\sigma'}(S)$, a contradiction. We deduce that $S = O_{\sigma,\sigma',\sigma}(S)$.

Let U be an α-invariant Hall σ'-subgroup of S. By the Frattini argument, $S = O_\sigma(S) \cdot N_S(U)$. Let V be an α-invariant Hall σ-subgroup of $N_S(U)$. Then

$$S = O_\sigma(S) \cdot U \cdot V. \tag{8}$$

If $[U, V] = 1$ then $O_\sigma(S) \cdot V \trianglelefteq S$ and so $S = O_{\sigma,\sigma'}(S)$, a contradiction. So

$$[U, V] \neq 1. \tag{9}$$

Let M_1, M_2 be two non-identity α-invariant normal subgroups of S such that $M_1 \cap M_2 = 1$. Then $S \lesssim (S/M_1) \times (S/M_2)$. As S/M_1 and S/M_2 have σ-length 1, S has σ-length 1, a contradiction. It follows that

$$S \text{ has a unique minimal } \alpha\text{-invariant normal subgroup.} \tag{10}$$

Thus, by (7), there is a $q \in \pi(S) \cap \sigma$ such that $F(S)$ is a q-group. Let $Q = F(S)$. If $\Phi(Q) \neq 1$ then $S/\Phi(Q)$ has σ-length 1. It follows that, $[U, V] \leqslant C_S(Q \bmod \Phi(Q))$. By Theorems 5.1.4 and 6.1.3 of [10], $[U, V] \leqslant C_S(Q) \leqslant Q$. So $[U, V] \leqslant U \cap Q = 1$, in contradiction with (9). We have proved that

$$\Phi(Q) = 1. \tag{11}$$

We have proved (i).

By (9), $V \nleqslant C_S(U)$. Let $X^\alpha = X \leqslant U$ and $Y^\alpha = Y \leqslant V$ such that $Y \leqslant N_S(X)$ and $Y \nleqslant C_S(X)$ and $|X|$ is minimal subject to the preceding conditions. By Theorem 5.3.6 of [10], $X = [X, Y]$. Let $S_0 = QXY$. If $S_0 \neq S$ then S_0 has σ-length 1. By Theorem 6.1.3 of [10], as $F(S)$ is a q-group, $O_{\sigma'}(S_0) = 1$. So $QY \trianglelefteq S_0$. But now $X = [X, Y] \leqslant QY$, a σ-group, a contradiction as $X \neq 1$. So $S = QXY$. We deduce that, as QY is a σ-group and X is a σ'-group,

$$S = QUV \qquad \text{and} \qquad U = [U, V]. \tag{12}$$

We have also proved the following two results.

(*) Let $Y^\alpha = Y \leqslant V$ and $X^\alpha = X \lneqq U$. Then, if $Y \leqslant N_S(X)$, $Y \leqslant C_S(X)$.

(**) Let $Y^\alpha = Y \leqslant V$ such that $QUY \neq S$. Then $Y \leqslant C_S(U)$.

Suppose that $C_Q(U) \neq 1$. By (11) and (12), $C_Q(U) \trianglelefteq S$. So $S/C_Q(U)$ has σ-length 1. But now, by (12), $U \leqslant C_S(Q \text{ mod. } C_Q(U))$. By Theorem 5.3.2 of [10], $U \leqslant C_S(Q)$. As $Q = F(S)$ is a σ-group, by Theorem 6.1.3 of [10] and (9), we have a contradiction. So

$$C_Q(U) = 1. \tag{13}$$

As $[Q \cap V, U] \leqslant Q \cap U = 1$, we deduce from (13) that

$$Q \cap V = 1. \tag{14}$$

Let $\hat{r} \in \pi(U)$ and $\hat{R}$ be a V-invariant Sylow r-subgroup of U. If $\hat{R} \neq U$ then, by (*), $\hat{R} \leqslant C_S(V)$. Therefore, by (9), $U = \hat{R}$. Thus U is an r-group for some $r \in \pi(S) \cap \sigma'$. By (*) and Theorem 5.3.8 of [10], U is a special r-group, $V \cdot \langle\alpha\rangle$ acts irreducibly on $U/\Phi(U)$ and $C_U(V) = \Phi(U)$. By (9) and (12), we have established (iii), (iv) and (vi).

Let $Y^\alpha = Y \lneqq V$. Then, by (14), $|QY| < |QV|$. So $QUY \neq S$. By (**), $Y \leqslant C_S(U)$. It follows that V is a t-group for some $t \in \pi(S) \cap \sigma$. We also have (v) and (viii).

If $q = t$ then, by (12), $QC_V(U) \trianglelefteq S$ and so, by (14), $C_V(U) \leqslant Q \cap V = 1$. We have (vii). We just have to prove (ii), (ix) and (x).

Let Q_0 be an irreducible $U \cdot V\langle\alpha\rangle$-subgroup of Q. If Q_0UV has σ-length 1 then, by (12),

$$U = [U, V] \leqslant O_{\sigma',\sigma}(Q_0UV).$$

So $U \trianglelefteq Q_0UV$. But now, by (13), $Q_0 \leqslant C_Q(U) = 1$, a contradiction. We deduce that $S = Q_0UV$, so that $Q_0V = QV$ and $Q = Q_0(V \cap Q)$. By (14) we have (ii).

If $C_Q(\alpha) = 1$ then, by what we have proved, S satisfies the hypotheses of 2.8 But now $U = 1$ in contradiction with (vi). We have (ix).

Also, by (iii) and (vi), $C_{\Phi(U)}(\alpha) \leqslant Z(U \cdot V\langle\alpha\rangle)$. So $C_Q(C_{\Phi(U)}(\alpha))$ is $U \cdot V \cdot \langle\alpha\rangle$-invariant. By (ii), Q is irreducible and so $C_Q(C_{\Phi(U)}(\alpha)) = 0$ or $C_Q(C_{\Phi(U)}(\alpha)) = Q$. Now, as F is nilpotent, $C_{\Phi(U)}(\alpha)$ centralises $C_Q(\alpha)$. By (ix), we see that $C_{\Phi(U)}(\alpha)$ centralises Q. By (ii), $C_{\Phi(U)}(\alpha) = 1$. We have (x) and the lemma.

We next obtain the main results of this section.

THEOREM 2.11. *Let t be a prime and $A \leqslant F_t$. Then $[O_{t'}(S), A] \leqslant F(S)$.*

Proof. Assume false and let S denote a minimal counterexample. As $O_{t'}(F(S)) = F(O_{t'}(S))$, $S = O_{t'}(S) \cdot A$. Now, by 2.2 and Theorem 6.1.3 of [10], as $[O_{t'}(S), A] \leqslant F(S)$,

$$A \leqslant C_S(F_2(O_{t'}(S) \bmod. F(O_{t'}(S)).$$

It follows that $O_{t'}(S) = F_2(O_{t'}(S))$.

Let $H^\alpha = H \leqslant S$. By Theorem 6.2.2 of [10] and Theorem 2.4.1 of [15], as F is nilpotent and $H/(H \cap O_{t'}(S)) = HO_{t'}(S)/O_{t'}(S) \leqslant AO_{t'}(S)/O_{t'}(S)$, $H = (H \cap O_{t'}(S)) \cdot O_t(C_H(\alpha))$. Clearly $O_t(C_H(\alpha)) = A \cap H$. By minimality of S, as $O_{t'}(S) \cap H \leqslant O_{t'}(H)$,

$$[O_{t'}(S) \cap H, A \cap H] \leqslant F(H).$$

It follows that $H = F_2(H)$.

Let $1 \neq X = X^\alpha \trianglelefteq S$. For $K \leqslant S$, let $\bar{K} = KX/X$. Then $\bar{A} \leqslant O_t$ $(\bar{S} \bmod. F(\bar{S}))$. So, as $O_{t'}(S) = F_2(O_{t'}(S))$, $\bar{S} = F_2(\bar{S})$. Assume that $q, r \in \pi(F(S))$ and $q \neq r$. Then, as $S \lesssim (S/O_q(S)) \times (S/O_r(S))$, S has Fitting height at most 2. But now $[O_{t'}(S), A] \leqslant F(S)$, a contradiction. So $\pi(F(S)) = \{q\}$, some prime q.

Set $\sigma = \{q, t\}$. Assume that S has σ-length 1. As $F(S)$ is a q-group, by Theorem 6.1.3 of [10], $O_{\sigma'}(S) = 1$. As $S = O_{t'}(S) \cdot A$ and $O_{t'}(S) = F_2(O_{t'}(S))$, $O_q(S) \in S_q(S)$ and $A \in S_t(S)$. But now $O_q(S) \cdot A \trianglelefteq S$. So $[O_{t'}(S), A] \leqslant O_{t'}(S) \cap (O_q(S) \cdot A) = O_q(S) = F(S)$, a contradiction. We deduce that S does not have σ-length 1.

As every α-invariant proper subgroup and every quotient of S by a non-identity α-invariant normal subgroup of S has Fitting height 2, all such have σ-length 1. So S satisfies the conclusion of 2.10. We use the notation and results of 2.10. We note that by 2.10 (x), $C_{\Phi(U)}(\alpha) = 1$.

Let $Z = GF(q)$ and K be a splitting field for $U \cdot V \cdot \langle\alpha\rangle$ which contains Z. Let $N = Q \otimes_Z K$ and M be an irreducible $K[U \cdot V \cdot \langle\alpha\rangle]$-submodule of N. By 2.4(ii), M is also faithful. By 2.4(iii), as F is nilpotent, $C_{UV}(\alpha)$ centralises $C_M(\alpha)$.

Suppose that U is not abelian. Then $\Phi(U) = Z(U)$ and $C_{Z(U)}(\alpha) = 1$. By 2.9, $C_{UV}(\alpha) = 1$. But now, by Theorem 10.2.1 of [10], $[U, V] = 1$, in contradiction with 2.10 (6). So U is abelian.

As $A \in S_t(S)$, $V = A$. By 2.10 (6), $C_U(V) = 1$. So, as F is nilpotent, $C_U(\alpha) \leqslant C_U(V) = 1$. By 2.9, we have that $C_{UV}(\alpha) = 1$. So $A = 1$. But now $[O_{t'}(S), A] = 1 \leqslant F(S)$, a contradiction.

So no minimal counterexample to the theorem can exist and the theorem is proved.

This next result is a generalisation of an unpublished result of G. Higman, using methods developed in unpublished work of N. K. Dickson.

THEOREM 2.12. *Let q be a prime such that F_q is abelian. Then S has q-length* 1.

Proof. Assume false and let S denote a minimal counterexample. Then the result of 2.10 hold for S with $\sigma = \{q\}$, so that $q = t$. We use the notation of 2.10. By 2.10(ix) and 2.10(x), $C_Q(\alpha) \neq 1$ and $C_{\Phi(U)}(\alpha) = 1$.

As F_q is abelian, $F_q \leqslant Z(F)$. It follows that $C_Q(\alpha)$ centralises $C_{UV}(\alpha)$. Let $Z = GF(q)$ and K be a splitting field for $U \cdot V \cdot \langle\alpha\rangle$ that contains Z. Let M be an irreducible $K[U \cdot V \cdot \langle\alpha\rangle]$-submodule of $Q \otimes_Z K$. By 2.4, M is also faithful. Also $C_{UV}(\alpha)$ centralises $C_M(\alpha)$.

If U is non-abelian, $\Phi(U) = Z(U)$. As $C_{\Phi(U)}(\alpha) = 1$, by 2.9, $C_{UV}(\alpha) = 1$. But, by 2.10 and Theorem 10.2.1 of [10], $C_{UV}(\alpha) \neq 1$. We deduce that U is abelian. By 2.9, it follows that $C_U(\alpha) \neq 1$. Suppose that $C_V(\alpha) \neq 1$. Then, by 2.10(vi), (vii), (viii), $C_U(\alpha) \leqslant C_U(V) = 1$, a contradiction. So we have proved:

$$U \text{ is abelian and } C_V(U) = C_V(\alpha) = 1. \tag{15}$$

Assume that there are p Wedderburn components of M with respect to UV. Let W be such a homogeneous component. Then

$$M = W \oplus W\alpha \oplus W\alpha^2 \oplus \cdots \oplus W\alpha^{p-1}. \tag{16}$$

So $C_M(\alpha) = W(1 + \alpha + \cdots + \alpha^{p-1})$. Let $w \in W$ and $x \in C_U(\alpha)^{\#}$. Then

$$wx(1 + \cdots + \alpha^{p-1}) = w(1 + \cdots + \alpha^{p-1})x = w(1 + \cdots + \alpha^{p-1}).$$

By (16), as $wx \in W$, $wx = w$. Using (16), we see that $C_{UV}(\alpha)$ acts trivially on M. By 2.10(ii), $C_{UV}(\alpha) = 1$, a contradiction. So M is a homogeneous $K[UV]$-module.

As in 2.9, the number of distinct isomorphism classes of irreducible U-submodules of M is a power of q. So there is an α-invariant Wedderburn component, W, with respect to U. Let $I(W) = \{g \in U \cdot V \cdot \langle\alpha\rangle : Wg = W\}$.

Now, by Lemma 3.2.1 of [10], as U is abelian, U is represented on W as a group of scalar matrices. If $W = M$ then, by 2.10(vi), $U = C_U(V) = 1$, a contradiction. So $I(W) \neq U \cdot V \cdot \langle\alpha\rangle$. As $U \cdot \langle\alpha\rangle \leqslant I(W)$, by 2.10(v), as $C_V(U) = 1$, $I(W) \cap V = 1$. Let $V = \{v_1, \ldots, v_n\}$ where $v_1 = 1$. Then

$$M = \bigoplus_{i=1}^{n} Wv_i \tag{17}$$

Each Wedderburn component of M with respect to U is a Wv_i, some i. As U is represented on W as a group of scalar matrices, $U \cdot \langle\alpha\rangle / C_{U\cdot\langle\alpha\rangle}(W)$ is abelian. By Theorem 3.2.3 of [10], an irreducible $U \cdot \langle\alpha\rangle$ submodule W_0 of W has dimension 1.

Let $M_0 = \bigoplus_{i=1}^{n} W_0 v_i$. Then M_0 is V-invariant. Also, for $1 \leqslant i \leqslant n$, $W_0 v_i \alpha = W_0 \alpha^{-1} v_i \alpha = W_0 v_i{}^\alpha \leqslant M_0$. So $M_0 \alpha \leqslant M_0$. Lastly, for $u \in U$ and $1 \leqslant i \leqslant n$, $W_0 v_i u = W_0(v_i u v_i^{-1}) v_i$. As $v_i u v_i^{-1} \in u$, $W_0 v_i u = W_0 v_i$. So M_0 is U-invariant. But now M_0 is $U \cdot V \cdot \langle\alpha\rangle$-invariant and so, as $M_0 \neq 0$, by irreducibility of M, $M = M_0$. We conclude that $W = W_0$.

Assume that, for some i, $Wv_i \alpha = Wv_i$. Then $v_i \alpha v_i^{-1} \in I(W)$. So $\alpha^{-1} v_i \alpha v_i^{-1} \in I(W) \cap V = 1$. So $v_i \alpha = v_i$ and $v_i \in C_V(\alpha) = 1$. So $Wv_i = W$. We deduce that W is the only α-invariant Wedderburn component of M with respect to U. Thus α permutes the remaining Wedderburn components in cycles of length p. Using (17), let $v \in V^{\#}$ and set

$$M_1 = Wv \oplus Wv\alpha \oplus \cdots \oplus Wv\alpha^{p-1}.$$

Then $C_{M_1}(\alpha) = Wv(1 + \cdots + \alpha^{p-1})$. Let $u \in C_U(\alpha)^{\#}$ and $x \in Wv$. (Recall that $C_U(\alpha) \neq 1$.) Then

$$x(1 + \cdots + \alpha^{p-1}) = x(1 + \cdots + \alpha^{p-1})u = xu(1 + \cdots + \alpha^{p-1}).$$

As $xu \in Wv$, $xu = x$. So u acts trivially on each Wedderburn component of M with respect to U except W. As $W = W_0$, $W = \langle w \rangle$ for some $w \in W$. Set $w_i = wv_i$, $1 \leqslant i \leqslant n$. Then $w_1, \ldots, w_n$ is a basis for M. Let $\lambda \in K$ such that $wu = \lambda w$. Then $\lambda \neq 1$ and u has matrix

$$\begin{bmatrix} \lambda & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix},$$

with respect to $w_1, \ldots, w_n$. Clearly $\det u = \lambda$. But 2.10(vi) yields that $U = [U, V] \leqslant G'$, so $\det u = 1$. This is a contradiction. So no minimal counterexample to the theorem can exist and the theorem is proved.

We now obtain a useful corollary.

COROLLARY 2.13. *Let q be a prime and $W \leqslant Z(F_q)$. Let B be an α-invariant q-subgroup of G such that $B = [B, W]$. Then $B \leqslant O_{q'}(F(S))$.*

Proof. By 2.11, it is enough to show that $B \leqslant O_{q'}(S)$. We may therefore suppose that $O_{q'}(S) = 1$. Using Theorem 5.1.4 of [10], we see that we can suppose that $\Phi(O_q(S)) = 1$. By Theorem 6.1.3 of [10], $C_S(O_q(S)) = O_q(S)$. So we can suppose that $S = O_q(S) \cdot B \cdot W$.

As $O_q(S) \cdot W \in S_q(S)$, by Theorem 6.2.2 of [10], $F_q \leqslant O_q(S) \cdot W$. Let $X = O_q(S) \cdot F$. By 2.3(ii), $F_q = XW$. As $O_q(S)$ is abelian and $W \leqslant Z(F_q)$, F_q is abelian. So, by 2.12, as $O_{q'}(S) = 1$, $W \leqslant O_q(S)$. But now

$$B = [B, W] \leqslant B \cap O_q(S) = 1 \leqslant O_{q'}(S).$$

The corollary is proved.

The next result could be made stronger if we assumed that p is not a Fermat prime.

LEMMA 2.14. *Let $q \in \pi(S) - \{2\}$ and Q be an α-invariant Sylow q-subgroup of S. Assume that $F \leqslant N_S(Q)$. Then $S = O_{q,q'}(S) \cdot F$.*

Proof. Assume that 2.14 is false and let S denote a minimal counterexample. Using Theorem 6.2.2 of [10], we find that $O_q(S) = 1$. Clearly $[Q, \alpha] \neq 1$. By Theorem 6.1.3 of [10], there is an $r \in \pi(F(S))$ such that $[Q, \alpha] \nleqslant C_S(O_r(S))$. Set $S_0 = O_r(S)[Q, \alpha]$. If $S_0 \neq S$ then, by Theorem 5.3.6 of [10], $[Q, \alpha] \leqslant O_q(S_0)$. But now $[Q, \alpha] \leqslant C_S(O_r(S))$, a contradiction. So $S = S_0$ and

$$S = O_r(S) \cdot Q \qquad \text{and} \qquad Q = [Q, \alpha]. \tag{18}$$

Using Theorem 5.1.4 of [10], we see that we may suppose $\Phi(O_r(S)) = 1$. Let $D = O_r(S) \cap C_S(Q)$. By (18), as $O_r(S)$ is abelian, $D \trianglelefteq S$. If $D \neq 1$ then, by minimality of S, $[Q, \alpha] \leqslant O_q$ (S mod. D). So $[Q, \alpha] \leqslant C_S(O_r(S)$ mod. $D)$. By Theorem 5.3.2 of [10], and (18), as $O_q(S) = 1$, $[Q, \alpha] = 1$, a contradiction. So

$$O_r(S) \cap C_S(Q) = 1. \tag{19}$$

As $F_r \leqslant N_S(Q)$, by (18), $[Q, F_r] \leqslant Q \cap O_r(S) = 1$. So, by (18) and (19), $F_r = 1$. As $O_q(S) = 1$, $Q \cdot \langle\alpha\rangle$ acts faithfully on $O_r(S)$. As $O_r(S) \cap F = 1$, by Theorem 3.1 and Corollary 3.2 of [24], as q is odd, $[Q, \alpha] = 1$. But now S is not a counterexample. The lemma is proved.

We require an easy lemma before proceeding.

LEMMA 2.15. *Assume that $F_q \leqslant O_q(S)$. Then S has q-length* 1.

Proof. By a standard argument using Theorem 5.1.4 of [10], S has q-length 1 if and only if $S/\Phi(O_q(S))$ has q-length 1. So we may suppose that $\Phi(O_q(S)) = 1$. But now F_q is abelian. By 2.12, S has q-length 1. The lemma follows.

LEMMA 2.16. *Assume that $O_q(S) \cap F$ is cyclic. Then S has q-length* 1.

Proof. Assume false and let S denote a minimal counterexample. Proceeding as in 2.15, we see that $\Phi(O_q(S)) = 1$. It follows from the main theorem of [18] that S has Fitting height at most 3. So $S = O_{q,q',q,q'}(S)$. It follows that $O_{q,q',q}(S)$ does not have q-length 1. So $S = O_{q,q',q}(S)$.

Let $U^\alpha = U$ be a Hall q'-subgroup of $O_{q,q'}(S)$ and $V^\alpha = V \in S_q(N_S(U))$. By the Frattini argument,

$$S = O_q(S) \cdot N_S(U) = O_q(S) \cdot U \cdot V. \tag{20}$$

By Theorem 6.2.2 of [10], $U = [U, V] \cdot C_U(V)$. Set $Q = O_q(S) \cdot V$. Then $Q \in S_q(S)$. Let $K = O_q(S) \cdot [U, V] \cdot V$. Suppose that $K \lneqq S$. Now $K \trianglelefteq K \cdot C_U(V) = O_q(S) \cdot U \cdot V = S$. Therefore, $O_q(K) \trianglelefteq S$, which yields that $O_q(K) = O_q(S)$.

By minimality of S, K has q-length 1. As $U = [U, V] \cdot C_U(V)$, $[U, V, V] = [U, V]$. So $[U, V] \leqslant O_{q'}(K) \leqslant C_S(O_q(K)) = C_S(O_q(S))$. By (20), $[U, V] \trianglelefteq S$, so that $[U, V] \leqslant O_{q'}(S)$.

For $H \leqslant S$, let $\bar{H} = HO_{q'}(S)/O_{q'}(S)$. Then $\bar{U}$ normalises $\overline{O_q(S)} \cdot \bar{V}$. So, as $\bar{S} = \bar{Q}\bar{U}$, $\bar{S}$ is q-closed. But now S has q-length 1, a contradiction. Thus $K = S$. So

$$U = [U, V]. \tag{21}$$

Let $D = O_q(S) \cap C_S(U)$. Suppose that $D \neq 1$. For $H \leqslant S$, set $\bar{H} = HD/D$. Then $O_q(\bar{S}) = \overline{O_q(S)}$. So, by minimality of S and Theorem 6.2.2 of [10], $\bar{S}$ has q-length 1. So, by (21), $\bar{U} = [\bar{U}, \bar{V}] \leqslant O_{q',q}(\bar{S})$. It follows that $\bar{U} \leqslant O_{q'}(\bar{S})$, so $[\overline{O_q(S)}, \bar{U}] = 1$. Thus $[O_q(S), U] \leqslant D$. But now $[O_q(S), U, U] = 1$. By Theorem 5.3.6 of [10], $[O_q(S), U] = 1$. By (20), $U \trianglelefteq S$. But now S has q-length 1, a contradiction. So

$$O_q(S) \cap C_S(U) = 1. \tag{22}$$

We have that $[O_q(S) \cap V, U] \leqslant O_q(S) \cap U = 1$. So, by (22),

$$O_q(S) \cap V = 1. \tag{23}$$

Let $X^\alpha = X \leqslant V$ and $S_X = O_q(S) \cdot U \cdot X$. If $S_X = S$ then $O_q(S) \cdot X = O_q(S) \cdot V$. So $V = X \cdot (O_q(S) \cap V)$. By (23), $V = X$. We note that $O_q(S_X)$ is U-invariant. So, by Theorem 6.3.2 of [10], $O_q(S_X) = O_q(S)$. Thus, if $S_X \lneqq S$, by minimality of S, S_X has q-length 1. So $[U, X] \leqslant O_{q'}(S_X)$.

We deduce that $[U, X] \leqslant C_S(O_q(S))$. But, by (20), $C_U(O_q(S)) = O_{q'}(S)$. So $[U, X] \leqslant O_{q'}(S)$. It follows that $X \leqslant O_q\ (S \bmod. O_{q'}(S)) = O_{q',q}(S)$. So we have proved that

(*) If $X^\alpha = X \lneqq V$ then $X \leqslant O_{q',q}(S)$.

Let $E = O_q(S) \cap F$. By 2.3(ii), $C_Q(\alpha) = E \cdot C_V(\alpha)$. If $C_V(\alpha) \leqslant O_{q',q}(S)$ then $F_q = C_Q(\alpha) \leqslant O_{q',q}(S)$. By Theorem 6.2.2 of [10] and 2.15 applied to $S/O_{q'}(S)$,

$S/O_{q'}(S)$ has q-length 1. But now S has q-length 1, a contradiction. So $C_V(\alpha) \nleqslant O_{q',q}(S)$.

Let $x \in C_V(\alpha) - O_{q',q}(S)$. Then, by (*), $V = \langle x \rangle$. Let $\langle z \rangle = O_q(S) \cap F$. As $\Phi(O_q(S)) = 1$, $\langle z \rangle \cong Z_q$ and $\langle z \rangle \trianglelefteq F_q$. As $F_q = C_Q(\alpha) = \langle z \rangle \langle x \rangle$, $F_q/Z(F_q)$ is cyclic. So F_q is abelian. By 2.12, S has q-length 1. So S is no counterexample. The lemma follows.

We now drop Hypothesis 2.7.

LEMMA 2.17. *Let a be an integer such that $a \geqslant 2$. Let E be an elementary abelian 2-group of rank a. Let X be a cyclic subgroup of* $\mathrm{Aut}(E)$ *such that* $|X| = 2^a - 1$ *and X acts regularly on $E^{\#}$, let $Y \leqslant \mathrm{Aut}(E)$ such that $|Y| = 3$ and $C_E(Y) \cong Z_2$. Suppose that Y normalises X. Then $a = 3$.*

Proof. Now $C_X(Y)$ normalises $C_E(Y) \cong Z_2$. As X acts regularly on $E^{\#}$, $C_X(Y) = 1$.

Let K denote a splitting field for XY that contains $GF(2)$. Let $V = E \otimes_{GF(2)} K$. Let N be an irreducible $K[XY]$-submodule of V. As XY acts irreducibly on E, by 2.4, XY acts faithfully on N. Also $C_N(x) = 0$ for each $x \in X^{\#}$ and $\dim_K C_N(Y) \leqslant 1$.

As $C_X(Y) = 1$, no element of X can be represented on N as a scalar matrix. It follows from Lemma 3.2.1 of [10] that N is not a homogeneous $K[X]$-module. By Theorem 3.4.1 of [10], there are three Wedderburn components of N with respect to X.

Let $y \in Y^{\#}$ and W be a homogeneous component of N with respect to X. Then $N = W \oplus Wy \oplus Wy^2$. It follows that $C_N(y) = W(1 + y + y^2) \cong W$ (as vector spaces).

So $\dim_K N = 3 \dim_K W = 3 \dim_K C_N(Y)$. As $N \neq 0$, $\dim_K C_N(Y) = 1$ and $\dim_K N = 3$. It follows from 2.4(i) that $M = N$. So $a = \dim_K N = 3$. The lemma is proved.

3. WEAK CLOSURE RESULTS

We introduce

HYPOTHESIS 3.1. G is a group, p is an odd prime and P is a Sylow p-subgroup of G. Also $W \leqslant Z(P)$ such that $W \trianglelefteq N(J(P))$. Finally, if $x \in N(W) - C(W)$, then $m([W, x]) \geqslant 2$. If $p = 3$ then we require that $m([W, x]) \geqslant 3$.

The first theorem of this section appears as (a) of Theorem 9.3 of [2]. It is also a corollary of Theorem 14.14, p. 46 of [22].

THEOREM 3.2. *Let G be a group, p be an odd prime and P be a Sylow p-subgroup of G. Let $W \leqslant Z(P)$ such that $W \trianglelefteq N(J(P))$, and $(p - 1) \nmid |N(W): C(W)|$. Then W is weakly closed in P with respect to G.*

We also require

THEOREM 3.3. *Assume Hypothesis* 3.1. *Let* $\pi = \pi(\mathrm{Aut}_G(W)) - \{2\}$. *Assume further that if* Y *is a* π*-subgroup of* $N(W)$ *then* $2 \notin \pi(\mathrm{Aut}_G(Y))$. *Then* W *is weakly closed in* P *with respect to* G.

THEOREM 3.4. *Assume Hypothesis* 3.1. *Assume further that* $N(W) = N(P) = N(J(P))$ *is a solvable group. Also assume that for no* $r \in \pi(\mathrm{Aut}_G(W)) - \{2\}$ *and no* 2*-element* $t \in G$ *is there an* r*-subgroup* R *of* G *such that the following four conditions are satisfied*

(i) $O_r(N(P)) \leqslant R \leqslant N(P)$.

(ii) $t \in N(R)$.

(iii) $t^2 \in N(W) - C(W)$.

(iv) $[R, t] \nleqslant C(W)$.

Then W *is weakly closed in* P *with respect to* G.

Assume that at least one of 3.2, 3.3 and 3.4 is false. Let G be a group of smallest order such that G is a counterexample to 3.2, 3.3 or 3.4.

It follows that no proper subgroup of G can violate these theorems. Also there is an odd prime p, a Sylow p-subgroup P of G and a subgroup $W \leqslant Z(P)$ such that $W \trianglelefteq N(J(P))$, W is not weakly closed in P with respect to G and G, P, W satisfy the hypotheses of one of 3.2, 3.3 and 3.4.

Set $M = N(W)$. Clearly M is a proper subgroup of G. We consider the set $\mathscr{H}$ which consists of p-subgroups, X, of H such that $X \geqslant W$ but $N(X) \nleqslant M$.

For a p-group Y, let

$$e(Y) = \max\{|A| : A \text{ is an abelian subgroup of } Y\}.$$

For $H_1, H_2 \in \mathscr{H}$ we say that $H_1 \prec H_2$ if one of the following holds.

(a) $e(H_1) < e(H_2)$.

(b) $e(H_1) = e(H_2)$ but $|J(H_1)| < |J(H_2)|$.

(c) $e(H_1) = e(H_2)$, $|J(H_1)| = |J(H_2)|$ but $|H_1| < |H_2|$.

After showing that $\mathscr{H} \neq \emptyset$, we consider a maximal member, H, of $\mathscr{H}$ under the partial ordering $\prec$. Let $C = C(Z(O_p(G)))$ and $\bar{G} = G/C$. We show that $\bar{G}$ and C satisfy Hypothesis I of [8], and also corollary 1 of [8] to get the structure of $\bar{G}$. An easy argument then shows that G cannot exist. This argument resembles the proof of Theorem 9.3 in [2].

LEMMA 3.5. $\mathscr{H} \neq \emptyset$.

Proof. As W is not weakly closed in P with respect to G, there is a $g \in G$ such that $W^g \leqslant P$ but $W^g \neq W$. Let $h = g^{-1}$ and $X = P^h \cap M$. If $N(X) \leqslant M$

then $N_{P^h}(X) \leqslant M \cap P^h = X$. But now $P^h = X \leqslant M$. So there is an $m \in M$ such that $P^{hm} = P$. As $W \trianglelefteq N(J(P))$, we have

$$hm \in N(P) \leqslant N(J(P)) \leqslant M.$$

So $g \in M$. But now $W^g = W$, a contradiction. Thus $N(X) \lneqq M$. Also $W \leqslant P^h \cap P = X$. So $X \in \mathscr{H}$. The lemma is proved.

LEMMA 3.6. *Let R be a p-subgroup of M. Then*

(i) *If $W \leqslant R$ then $W \leqslant Z(R)$.*

(ii) *If $R \in S_p(M)$ then $N(R) \leqslant N(J(R)) \leqslant M$.*

Proof. Now there is an $m \in M$ such that $R^m \leqslant P$. Suppose that $W \leqslant R$. Then

$$W = W^m \leqslant R^m \cap Z(P) \leqslant Z(R^m) = Z(R)^m.$$

We deduce that (i) holds.

Suppose that $R \in S_p(M)$. Then $R^m = P$. So, as $W \trianglelefteq N(J(P))$,

$$N(J(R^m)) = N(J(R))^m = N(J(P)) \leqslant M.$$

We conclude that $N(J(R)) \leqslant M$. As $N(R) \leqslant N(J(R))$, we have (ii) and the lemma.

Let H be a maximal member of $\mathscr{H}$ under the partial ordering $\prec$.

LEMMA 3.7. *Let R be a p-subgroup of M such that $R \gneqq H$. Then $N(R) \leqslant M$. Furthermore, if $J(R) \nleqslant H$, then $N(J(R)) \leqslant M$.*

Proof. Suppose that $N(R) \nleqslant M$. Then $R \in \mathscr{H}$. So, by maximality of H, $e(R) \leqslant e(H) \leqslant e(R)$. So $e(H) = e(R)$ and $J(H) \leqslant J(R)$. But, by maximality of $\mathscr{H}$, $|J(H)| \geqslant |J(R)|$ so $J(H) = J(R)$. But $|H| < |R|$, in contradiction with the maximality of H. So $N(R) \leqslant M$.

Suppose that $J(R) \nleqslant H$ and that $N(J(R)) \leqslant M$. By 3.6(i), $W \leqslant Z(R) \leqslant J(R)$. So $J(R) \in \mathscr{H}$. Thus

$$e(R) = e(J(R)) \leqslant e(H) \leqslant e(R).$$

So $e(H) = e(R)$ and $J(H) \leqslant J(R)$. But, by maximality of H,

$$|J(J(R))| = |J(R)| \leqslant |J(H)|.$$

We conclude that $J(H) = J(R) \leqslant H$, a contradiction so, if $J(R) \nleqslant H$, $N(J(R)) \leqslant M$. The lemma follows.

We now establish the main minimality argument. Let E denote the core of $O_{p'}(N(P))$ in M (largest normal subgroup of M contained in $O_{p'}(N(P))$).

LEMMA 3.8. *Let K be a proper subgroup of G such that $K \nleqslant M$, $W \leqslant O_p(K)$ and $E \leqslant K$. Assume that there is a Sylow p-subgroup Q of $K \cap M$ such that $Q \in S_p(K)$. Then $N_K(J(Q)) \nleqslant M$.*

Proof. Suppose that $N_K(J(Q)) \leqslant M$. We note that $W \leqslant O_p(K) \leqslant Q$. By 3.6(i), $W \leqslant Z(Q)$. So $W \trianglelefteq N_K(J(Q))$. We now prove

(*) *W is weakly closed in Q with respect to K.*

For any subgroup X of K, we observe that, as $C_K(X) = N_K(X) \cap C(X)$,

$$| N_K(X) : C_K(X)| = | N_K(X)\, C(X) : C(X)| \text{ which divides } | N(X) : C(X)| \qquad (1)$$

Suppose that G is a counterexample to 3.2. Then, by (1), $(p - 1)$ does not divide $| N_K(W) : C_K(W)|$. By minimality of G, (*) holds.

Now suppose that G is a counterexample to 3.3. As $N_K(W) - C_K(W) \subseteq N(W) - C(W)$, K, Q, W satisfy Hypothesis 3.1. Let $\pi^* = \pi(\mathrm{Aut}_K(W)) - \{2\}$. Let Y be a π^*-subgroup of $N_K(W)$

By (1), $\pi^* \subseteq \pi$ and $| \mathrm{Aut}_K(Y)|$ divides $| \mathrm{Aut}_G(Y)|$ But now Y is a π-subgroup of $N(W)$ and so $2 \notin \pi(\mathrm{Aut}_G(Y))$. So $2 \notin \pi(\mathrm{Aut}_K(Y))$. Again, by minimality of G, (*) holds in this case.

Lastly suppose that G is a counterexample to 3.4. As with 3.3, K, Q, W satisfy Hypothesis 3.1. Also $Q \trianglelefteq K \cap M$. As $N_K(J(Q)) \leqslant M$, we have that $N_K(Q) = N_K(J(Q)) = N_K(W) \leqslant M$, a solvable group.

Let $\pi^* = \pi(\mathrm{Aut}_K(W)) - \{2\}$. Suppose that there is an $r \in \pi^*$ and an r-subgroup R of K such that, for some 2-element $t \in K$,

(i) $O_r(N_K(Q)) \leqslant R \leqslant N_K(Q)$

(ii) $t \in N_K(R)$

(iii) $t^2 \in N_K(W) - C_K(W)$

(iv) $[R, t] \nleqslant C_K(W)$

As G is a counterexample to 3.4, $E = O_{p'}(N(P))$. As $E \leqslant K \cap M$, $O_r(N(P)) \leqslant O_r(N_K(Q)) \leqslant R \leqslant N_K(Q) \leqslant N(P)$. But now, as $N_K(W) - C_K(W) \subseteq N(W) - C(W)$, r, R and t satisfy conditions (i), (ii), (iii), and (iv) for G, P, W. By (i), $\pi^* \subseteq \pi(\mathrm{Aut}_G(W)) - \{2\}$. This is a contradiction. So no such r, R can exist and, by minimality of G, we have (*). We deduce that, in all cases, (*) holds.

Now $K \nleqslant M$. Let $k \in K - M$. As $W \leqslant O_p(K) \leqslant Q$,

$$W^k \leqslant O_p(K) \leqslant Q.$$

But, by (*), $W^k = W$. So $k \in M$, a contradiction. The lemma is proved.

LEMMA 3.9. $J(H) \trianglelefteq G$.

Proof. Let $L = N(H)$ and $R \in S_p(L \cap M)$. Suppose that $R = H$. Then $H \in S_p(N_M(H))$ and so $H \in S_p(M)$. But now, by 3.6(ii), $N(H) \leqslant M$, a contradiction. So

$$R \gneqq H. \tag{2}$$

By 3.7, $N(R) \leqslant M$. So $N_L(R) \leqslant L \cap M$. Thus $R \in S_p(N_L(R))$ and so

$$R \in S_p(L). \tag{3}$$

We now consider two cases as $N(J(R)) \nleqslant M$ and $N(J(R)) \leqslant M$.

Case I. $N(J(R)) \nleqslant M$. By 3.7, $J(R) \leqslant H$. So $e(R) = e(J(R)) \leqslant e(H)$. As $H \leqslant R$, $e(H) \leqslant e(R)$. Thus $e(H) = e(R)$. As $H \leqslant R$, we have that $J(H) = J(R)$.

Let $R^* \in S_p(N(J(R)))$, $R^* \geqslant R$. Then $R^* \cap M \geqslant R \gneqq H$. So, by 3.7, $N_{R^*}(R^* \cap M) \leqslant R^* \cap M$. We deduce that $R^* \leqslant M$.

Suppose that $J(R^*) \leqslant H$. Then, since $H \lneqq R \leqslant R^*$,

$$e(R^*) \leqslant e(H) \leqslant e(R) \leqslant e(R^*).$$

Thus

$$J(R^*) = J(H) \leqslant J(R) \leqslant J(R^*).$$

So $J(H) = J(R) = J(R^*)$. So $N(R^*) \leqslant N(J(R^*)) = N(J(R))$. As $R^* \in S_p(N(J(R)))$, $R^* \in S_p(N(R^*))$ and so $R^* \in S_p(G)$. As $R^* \leqslant M$, by 3.6(ii),

$$N(H) \leqslant N(J(H)) = N(J(R^*)) \leqslant M,$$

a contradiction. Thus $J(R^*) \nleqslant H$. By 3.7, $N(J(R^*)) \leqslant M$.

Let $L^* = N(J(R))$. Suppose that $L^* \lneqq G$. As $N(H) \nleqslant M$ and $J(H) = J(R)$, $L^* \nleqslant M$. As $W \leqslant H$, by 3.6(i),

$$W \leqslant Z(H) \leqslant J(H) = J(R) \leqslant O_p(L^*).$$

Now $R^* \in S_p(L^* \cap M)$ and $R^* \in S_p(L^*)$. Also $E \leqslant C(J(R)) \leqslant L^*$. So, by 3.8, $N_{L^*}(J(R^*)) \nleqslant M$, a contradiction. So $L^* = G$ and $J(H) \trianglelefteq G$.

Case II. $N(J(R)) \leqslant M$. Now, by (3), $R \in S_p(L \cap M)$ and $R \in S_p(L)$. Clearly $L \nleqslant M$. Also $W \leqslant H \leqslant O_p(L)$ and $E \leqslant C(H) \leqslant L$. By 3.8, $N_L(J(R)) \nleqslant M$, a contradiction.

The lemma is proved.

LEMMA 3.10. *There is an abelian subgroup A of P such that $|A| = e(P)$ and $A \nleqslant O_p(G)$. For such a subgroup A and any element $g \in G - M$, $\langle A, E, J(H), g\rangle = G$.*

Proof. Suppose first that no such A exists. Then $J(P) \leqslant O_p(G)$. But now $J(P) = J(O_p(G)) \trianglelefteq G$. As $W \trianglelefteq N(J(P))$, we have a contradiction.

Pick such an A. Let $g \in G - M$ and $L = \langle A, E, J(H), g \rangle$. Let $R \in S_p(L \cap M)$ such that $R \geqslant AJ(H)$. Then

$$e(R) = |A| = e(P).$$

Suppose that $R \notin S_p(L)$. Let $R^* \in S_p(L)$, $R^* \geqslant R$. Then $R \lneqq N_{R^*}(R)$, so that $N(R) \nleqslant M$. Since $W \leqslant J(H) \leqslant R$, $R \in \mathscr{H}$. So, by maximality of H, $e(H) \geqslant e(R)$, so that

$$e(H) = e(R).$$

Thus $J(H) \leqslant J(R)$. But $|J(R)| \leqslant |J(H)|$. Thus $J(H) = J(R)$. But, by 3.9, $J(H) \leqslant O_p(G)$. So $A \leqslant J(R) = J(H) \leqslant O_p(G)$, in contradiction with the choice of A. Thus $R \in S_p(L)$.

Suppose that $N(J(R)) \nleqslant M$. As $W \leqslant A \leqslant J(R)$, $J(R) \in \mathscr{H}$. So $e(H) = e(J(R)) = e(R) = |A| = e(P)$. Thus $e(H) = e(R)$. As $R \geqslant J(H)$, $J(H) \leqslant J(R)$. As $|J(R)| = |J(J(R))| \leqslant |J(H)|$, $J(H) = J(R)$. But $J(H) \leqslant O_p(G)$. So $A \leqslant J(R) = J(H) \leqslant O_p(G)$, a contradiction. Thus $N(J(R)) \leqslant M$.

Also, as $g \notin M$, $L \nleqslant M$. Now $W^g \leqslant J(H) \leqslant R$ and $W^g \neq W$. So W is not weakly closed in R with respect to L. So, by 3.8, $L = G$. The lemma follows.

Corollary 3.11. *M is a maximal subgroup of G.*

Proof. $\langle AJ(H), E \rangle \leqslant M$. So 3.11 follows.

Lemma 3.12. *Let $V = O_p(G)$, $C = C(Z(V))$ and $X = Z(V)/(Z(V) \cap Z(G))$. Then C/V is a p'-group, X is an elementary abelian group and G/C acts faithfully on X. Moreover*

(i) *There is an abelian subgroup A of P such that $|A| = e(P)$ and $A \nleqslant O_p(G)$.*

(ii) *There is a field K of endomorphisms of X such that X is a vector space of dimension 2 over K and the group of automorphisms of X induced by G is $SL(2, K)$.*

(iii) *There is an integer n such that $G/C \simeq SL(2, p^n)$ and $|X| = p^{2n}$.*

(iv) $Z(V) = (Z(V) \cap Z(G)) \times [Z(V), G]$.

Proof. Now $C \trianglelefteq G$. For $K \leqslant G$, let $\bar{K} = KC/C$. By 3.6(i), $W \leqslant Z(H) \leqslant J(H)$. By 3.9, $W \leqslant O_p(G) \leqslant P$. So $W \leqslant Z(V)$ and $C \leqslant C(W) \leqslant M$. By 3.10, $\bar{M}$ is a maximal subgroup of $\bar{G}$. Clearly $\bar{P} \in S_p(\bar{M})$.

Choose $A \leqslant P$ in accordance with 3.10. Then $\bar{A} \neq 1$ and $\bar{A}$ is abelian. Also, by 3.10, as $EJ(H) \leqslant C$, if $\bar{g} \in \bar{G} - \bar{M}$ then $\bar{G} = \langle \bar{A}, \bar{g} \rangle$. In order to verify Hypothesis I of [8] for $\bar{G}, \bar{P}, \bar{A}, \bar{M}$ and $Z(V)$, we just have to verify that $O_p(G/C) = 1$ and $[Z(V), A, A] = 1$.

Let $D = O_p(G \bmod. C)$ and $Q = P \cap D$. By the Frattini argument, $G = DN(Q) = CQN(Q) = CN(Q)$. As $C \leqslant M$, $N(Q) \nleqslant M$. Also, as $Q \leqslant P$, $P \leqslant N(Q)$ and $E \leqslant N(Q)$. So $A \leqslant N(Q)$ and, by 3.9, $J(H) \leqslant P$. So, by 3.10, $N(Q) = G$. So $Q \leqslant V \leqslant C$ and $D = C$. Also $V \in S_p(C)$. So

$$C/V \text{ is a } p'\text{-group and } O_p(G/C) = 1. \tag{4}$$

Choose an abelian subgroup A of P such that $|A| = e(P)$, $A \nleqslant V$ and $|A \cap Z(V)|$ is maximal subject to the preceding conditions. Suppose that $[Z(V), A, A] \neq 1$. By Theorem 4 of [8], there is an abelian subgroup A^* of A such that $|A^*| = e(P)$, $A \cap Z(V) \lneqq A^* \cap Z(V)$ and $[A^*, A, A] = 1$. By choice of A, $A^* \leqslant V$. Thus $Z(V)A^*$ is abelian. So

$$|Z(V)A^*| = |Z(V)|\,|A^*|/|Z(V) \cap A^*| \leqslant e(P) = |A^*|.$$

So $|Z(V)| \leqslant |Z(V) \cap A^*|$. Thus $Z(V) \leqslant A^*$ and $[Z(V), A, A] = 1$, a contradiction. So

$$[Z(V), A, A] = 1. \tag{5}$$

We conclude that $\bar{G}$, $\bar{P}$, $\bar{M}$, $\bar{A}$ and $Z(V)$ satisfy Hypothesis I. Let $A_0 = A \cap C$. Then $A_0Z(V)$ is abelian and $C_P(A) = A$. So

$$|A| \geqslant |A_0Z(V)| = |A_0|\,|Z(V)|/|A_0 \cap Z(V)| = |A_0|\,|Z(V)|/|C_{Z(V)}(A)|.$$

Thus

$$|AC/C| = |A|/|A_0| \geqslant |Z(V)|/|C_{Z(V)}(A)|.$$

By corollary 1 of [8], as p is odd, the lemma follows.

Proof of 3.2, 3.3 *and* 3.4. We adopt the notation of 3.12. For $L \leqslant G$, let $\bar{L} = LC/C$ and $\tilde{L} = L(Z(V) \cap Z(G))/(Z(V) \cap Z(G))$. By the Frattini argument, $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$. Then $\tilde{W}$ admits $N_{\bar{G}}(\bar{P})$. As $M \lneqq G$, $W \ntrianglelefteq G$ and so $\tilde{W} \neq 1$. By 3.12(iii), $\bar{G} \cong SL(2, p^n)$.

As $W \leqslant Z(P)$, $\tilde{W} \leqslant C_X(\bar{P})$. Regard X as a 2-dimensional vector space over K. We can suppose that $\bar{P} = \{\left[\begin{smallmatrix}1 & 0\\ a & 1\end{smallmatrix}\right]: a \in K\}$ and $N_{\bar{G}}(\bar{P}) = \bar{P}\bar{D}$ where $\bar{D} = \{\left[\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}\right]: a \in K - \{0\}\}$ with respect to some basis $\tilde{v}_1$, $\tilde{v}_2$ of X.

Note that $\langle \tilde{v}_1 \rangle = C_X(\bar{P})$ and that $W \leqslant \langle \tilde{v}_1 \rangle$. As $\tilde{W}$ admits $\bar{D}$,

$$\tilde{W} = C_X(\bar{P}) = \langle \tilde{v}_1 \rangle.$$

Let D be the pre-image in G of $\bar{D}$. Then $D \leqslant N(W)$. As $W \neq 1$ and $C_{\bar{D}}(\tilde{W}) = 1$, $C_D(W) = C$. As $p^n - 1 = |D : C|$, $(p^n - 1)$ divides $|N(W) : C(W)|$. But now $(p - 1)$ divides $|N(W) : C(W)|$.

We deduce that G, P, W do not satisfy the hypotheses of 3.2. So G, P, W satisfy Hypothesis 3.1. Now $D > C(W)$. Let $h \in D - C(W)$. Then $m([W, h]) \geqslant 2$ and, if $p = 3$, $m([W, h]) \geqslant 3$. So $n \geqslant 2$ and, if $p = 3$, $n \geqslant 3$. It follows that $p^n - 1$ is not a power of 2.

So there is an odd prime q, a Sylow q-subgroup Q of H and some $b \in K$ such that $\bar{Q} = \langle \left[\begin{smallmatrix} b^{-1} & 0 \\ 0 & b \end{smallmatrix}\right] \rangle \in S_q(\bar{D})$. Now $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in N_{\bar{G}}(\bar{D})$. Let N be the pre-image in G of $N_{\bar{G}}(\bar{D})$. Then $N = DN_N(Q)$. Let t be a 2-element of $N_N(Q)$ such that $\bar{t} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Then $\bar{t}$ inverts $\bar{Q}$. Also $\bar{t}^2 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in N_{\bar{G}}(\bar{P})$. Thus $t^2 \in N(W) - C(W)$. Also $t \in N(Q) - C(Q)$.

As $q \in \pi(D/C)$ and $D \leqslant N(W)$, $q \in \pi(\mathrm{Aut}_G(W)) - \{2\}$. As $2 \in \pi(\mathrm{Aut}_G(Q))$, G cannot satisfy the hypotheses of 3.3. So G, P, W satisfy the hypothesis of 3.4. We deduce that $E = O_{p'}(N(P))$.

Also $E \leqslant D \leqslant M$. Thus $O_q(N(P)) \leqslant O_q(D) \leqslant Q \leqslant N(P)$. As $\bar{t}$ inverts $\bar{Q}$, $\bar{Q} = [\bar{Q}, \bar{t}]$. So $[Q, t] \nleqslant C$. As $W \leqslant V \leqslant P$, $W \leqslant Z(V)$. So, as $[Q, t] \leqslant D$ and $C_D(W) = C$, $[Q, t] \nleqslant C(W)$. But now q, Q, t satisfy (i), (ii), (iii), and (iv) of 3.4. So G cannot be a counterexample to 3.4.

It follows that G cannot exist. So 3.2, 3.3 and 3.4 are proved.

4. Preliminary Results

In this section we will discuss the properties of weakly closed subgroups of Sylow subgroups of groups. We will also need to discuss groups which satisfy the following hypothesis.

Hypothesis 4.1. G is a group such that $3 \in \pi(G)$. Let P be a Sylow 3-subgroup of G. Assume that $N(P)$ is solvable and $N(P)/PC(P)$ is nilpotent.

Assume further that there is a set of primes, π, and a Hall π-subgroup U of $N(P)$ such that $3 \notin \pi$ and, if $\mathscr{X} = \{X \leqslant U : C_P(X) \neq 1\}$, there is a subgroup W_X, defined for each $X \in \mathscr{X}$, such that the following holds.

(A1) $W_X \leqslant \Omega_1(Z(C_P(X)))$.

(A2) If $g \in G$ such that $(W_X)^g \leqslant P$ then $g \in N(P)$.

(A3) If $n \in N(P)$ such that $X^n \leqslant U$ then $(W_X)^n = W_{X^n}$.

(A4) If $Y \leqslant N_U(X)$ and $u \in N_U(X) \cap N_U(Y)$ then either $m([C_{W_X}(Y), u]) \geqslant 3$ or u centralises $C_{W_X}(Y)$.

In Lemma 4.8 below we will discuss how the above hypothesis can arise. We now state what the principal results of this section are. We observe that, as $1 \in \mathscr{X}$, W_1 is defined.

Theorem 4.2. *Assume Hypothesis* 4.1. *Set* $W = W_1$. *Assume further that*

(B1) $2 \in \pi$.

(B2) $N(P) = PC(P)U$.

(B3) *If* $T \in S_2(U)$ *then* $N(C_T(P)) \leqslant N(P)$.

(B4) *Eeach* W*-invariant* $3'$*-subgroup of* G *lies in* $C(W)$.

Then $P \trianglelefteq G$.

THEOREM 4.3. *Assume Hypothesis* 4.1. *Assume further that*

(C1) $2 \in \pi$.

(C2) $N(P) = PC(P)U$.

(C3) $C(a) \leqslant N(P)$ *for each* $a \in P^{\#}$.

(C4) *If* $q \in \pi$ *and* V *is a* q-*subgroup of* U, *then* $O^q(N(V)) \leqslant C(V)$.

Then $G = O_{3'}(G) \cdot N(P)$.

THEOREM 4.4. *Assume Hypothesis* 4.1. *Set* $W = W_1$. *Assume further that*

(D1) $\pi = \{2\}$.

(D2) $N(P) = PC(P)U$.

(D3) *Each* W-*invariant* $3'$-*subgroup of* G *lies in* $C(W)$.

(D4) *If* $V \leqslant U$ *then* $O^2(N(V)) \leqslant C(V)$.

Then $P \trianglelefteq G$.

We now discuss the promised results on weak closure, and other well-known results. The first of these is Lemma 2 of [14].

LEMMA 4.5. *Let* G *be a group and* p *be a prime. Assume that* G *is* p-*solvable and that* P *is a* p-*subgroup of* G. *Then* $O_{p'}(N(P)) \leqslant O_{p'}(G)$.

LEMMA 4.6. *Let* G *be a group and* p *be a prime. Let* P *be a Sylow* p-*subgroup of* G *and* W *be a weakly closed subgroup of* P *with respect to* G. *Then the following holds.*

(i) *If* X *is a* p-*subgroup of* G *such that* $X \geqslant W$ *then* $N(X) \leqslant N(W)$ *and* W *is weakly closed in* X *with respect to* G. *Furthermore if* $W \leqslant Z(P)$ *then* $W \leqslant Z(X)$.

(ii) *If* x, y *are elements of* $C(W)$ *that are conjugate in* G, *then they are conjugate in* $N(W)$.

(iii) *If* X, Y *are subgroups of* $C(W)$ *that are conjugate in* G *then they are conjugate in* $N(W)$.

Proof. Let X be a p-subgroup of G such that $X \geqslant W$. Then there is a $g \in G$ such that $X^g \leqslant P$. Let $h = g^{-1}$. Let $k \in G$ such that $W^k \leqslant X$. Then W^{kg}, $W^g \leqslant X^g \leqslant P$. By weak closure of W, $k = (kg)h \in N(W)$. So $W^k = W$ and W is weakly closed in X with respect to G. Clearly now $N(X) \leqslant N(W)$.

Suppose $W \leqslant Z(P)$. As $W^g \leqslant P$, $g \in N(W)$. So $h \in N(W)$ and $W = W^h \leqslant Z(P^h)$. But $X \leqslant P^h$. So $W \leqslant Z(X)$. We have (i).

Let X, Y be non-empty subsets of $C(W)$ and $g \in G$ such that $X^g = Y$. Let $h = g^{-1}$. The $W \leqslant C(X) \cap C(X^g)$, so that W, $W^h \leqslant C(X)$, a subgroup of G.

Let $Q \in S_p(C(X))$ such that $Q \geqslant W$. Then there is a $c \in C(X)$ such that $W^{hc} \leqslant Q$. By (i), $W = W^{hc}$. So $hc \in N(W)$. Set $d = c^{-1}$. Then $dg \in N(W)$. But $Y = X^g = X^{dg}$. We have (ii), (iii) and the lemma.

LEMMA 4.7. *Let p, q, r be distinct primes. Let A be a p-group, B be a q-group and R be an r-group. Suppose that $d(A) \geqslant 2$ and that $A \times B$ acts as a group of operators on R in such a way that $R = [R, B]$. Then $R = \langle [C_R(a), B]: a \in A^{\#} \rangle$.*

Proof. Using Theorems 5.1.1 and 6.2.2 of [10], we find that we may suppose $\Phi(R) = 1$. As $R = [R, B]$, by Theorem 5.2.3 of [10], $C_R(B) = 1$. But, by Theorem 6.2.4 of [10],

$$R = \langle C_R(a): a \in A^{\#} \rangle.$$

Let $a \in A^{\#}$. As B centralises A, B normalises $C_R(a)$. As $C_R(B) = 1$, $C_R(a) = [C_R(a), B]$. The lemma follows.

The conditions for the following lemma are derived from situations that occur in Section 5.

LEMMA 4.8. *Let G be a group and p be a prime, $p \geqslant 5$. Assume that $3 \in \pi(G)$ and G admits an automorphism α of order p such that $F = C_G(\alpha)$ is a nilpotent p'-group. Let P be an α-invariant Sylow 3-subgroup of G, $Z = \Omega_1(Z(P))$ and $K = N(P) \cap F$. Assume further that the following holds.*

(a) *$N(P) = PC(P)K = N(Z)$.*

(b) *$N(P)$ is a solvable group.*

(c) *Z is weakly closed in P with respect to G.*

(d) *If Q is a non-identity α-invariant subgroup of P such that $Q \geqslant C_Z(\alpha)$ and $Z(K) \leqslant N(Q)$, then $N(Q) \leqslant N(P)$.*

Set $\pi = \pi(N(P)/PC(P))$. Let U be an α-invariant Hall π-subgroup of $N(P)$. Let $\mathscr{X} = \{X \leqslant U: C_P(X) \neq 1\}$. For $X \in \mathscr{X}$, set $W_X = \Omega_1(Z(C_P(X)))$. Then G, P, π, U and W_X satisfy Hypothesis 4.1. Furthermore $N(P) = PC(P)U$.

Proof. By 2.3(i), as p does not divide $|F|$, p does not divide $|G|$. By (a) and (b), $N(P) = PC(P)U$, $N(P)$ is solvable and $N(P)/PC(P)$ is nilpotent. We therefore just have to verify (A2), (A3) and (A4) of Hypothesis 4.1.

Let $X \leqslant U$. By (a), Theorem 2.4.1 of [15] and Theorem 6.2.2 of [10], as $U/C_U(P) = UPC(P)/PC(P)$, which is fixed by α,

$$U = C_U(P) \cdot C_U(\alpha). \tag{1}$$

For $X \leqslant U$, let $X^* = (C_U(P)X) \cap C_U(\alpha)$. Then, by (1), $C_U(P)X = C_U(P)X^*$. So $C_P(X) = C_P(X^*)$. We deduce that, if $C_P(X) \neq 1$, W_X is α-invariant and $Z(K)$-invariant. Also, by Theorem 6.2.2 of [10], $O_3(F) \leqslant P$ and so, as F is nilpotent, $C_Z(\alpha) \leqslant Z(O_3(F)) \leqslant Z(F)$. Thus $C_Z(\alpha) = C_P(X^*)$. We conclude that $C_Z(\alpha) \leqslant W_X$. By (d), $N(W_X) \leqslant N(P)$.

Let $g \in G$ such that $(W_X)^g \leqslant P$. By (a), (c) and 4.6(iii), there is an $n \in N(P)$ such that $(W_X)^g = (W_X)^n$. So $gn^{-1} \in N(W_X) \leqslant N(P)$, and $g \in N(P)$. We have (A2).

Let $X \in \mathscr{X}$ and $n \in N(P)$ such that $X^n \leqslant U$. Then $(W_X)^n = \Omega_1(Z(C_P(X)))^n = \Omega_1(Z(C_P(X)^n)) = \Omega_1(Z(C_P(X^n))) = W_{X^n}$. We have (A3).

Let $u \in U$. Then, by (1), $C_U(P)u = C_U(P)u^*$ for some $u^* \in C_U(\alpha)$. Suppose that $X \in \mathscr{X}$, $Y \leqslant N_U(X)$ and $u \in N_U(X) \cap N_U(Y)$. Now as with X^*, let $Y^* = (C_U(P)Y) \cap C_U(\alpha)$. Then $C_P(Y) = C_P(Y^*)$. Let $W = W_X$ and $D = [C_W(Y), u]$. So $D = [C_W(Y), u^*]$ is an α-invariant subgroup of $C_W(Y)$.

As F is nilpotent and $3 \notin \pi$, $C_D(\alpha) \leqslant C_D(u^*)$. By Theorem 5.2.3 of [10], as W is abelian, $C_D(u^*) = C_D(\alpha) = 1$. By the hypothesis of this lemma, $p \geqslant 5$. So $D = 1$ or $m(D) \geqslant 3$. We have (A4). The lemma is proved.

We now prove some technical lemmas which are used to find the structure of a minimal counterexample to 4.2, 4.3 or 4.4.

LEMMA 4.9. *Assume Hypothesis* 4.1. *Let q be a prime and H be a normal q-subgroup of G such that $H \leqslant N(P)$, $P \nleqslant H$ and $H \cap P \leqslant Z(G)$. For any subgroup K of G, set $\bar{K} = KH/H$.*

For $\bar{X} \leqslant \bar{U}$, let $V_{\bar{X}}$ be the pre-image of $\bar{X}$ and $S_{\bar{X}} = U \cap V_{\bar{X}}$. If $C_{\bar{P}}(\bar{X}) \neq 1$, let $W_{\bar{X}} = \overline{W_{S_{\bar{X}}}}$. Then the following holds.

(i) *Hypothesis* 4.1 *holds for $\bar{G}$ with P replaced by $\bar{P}$, U replaced by $\bar{U}$, W_X replaced by $W_{\bar{X}}$ and π as before. Also $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$.*

(ii) *Let $W = W_1$ (note $1 \in \mathscr{X}$). Then, if each W-invariant $3'$-subgroup of G lies in $C(W)$, each $\bar{W}$-invariant $3'$-subgroup of $\bar{G}$ lies in $C_{\bar{G}}(\bar{W})$.*

Proof. We must firstly check that $W_{\bar{X}}$ is well-defined. Let $\bar{X} \leqslant \bar{U}$, $V = V_{\bar{X}}$ and $S = S_{\bar{X}}$. Then $V = HS$.

Let R be the pre-image of $C_{\bar{P}}(\bar{X})$. Then $R = HQ$, where $Q = R \cap P$. Clearly $[Q, S] \leqslant P$. But

$$[\bar{Q}, \bar{S}] = [\overline{HQ}, \overline{HS}] = [C_{\bar{P}}(\bar{X}), \bar{X}] = 1.$$

So $[Q, S] \leqslant P \cap H \leqslant Z(G)$. Also $[Q, S] \leqslant P \cap H \leqslant P \cap R \leqslant Q$. We deduce that S normalises Q and $[Q, S, S] = 1$. By Theorem 5.3.6 of [10], as $3 \notin \pi$, $[Q, S] = 1$. So

$$C_{\bar{P}}(\bar{X}) = \bar{R} = \bar{Q} \leqslant \overline{C_P(S)} \leqslant C_{\bar{P}}(\bar{S}) = C_{\bar{P}}(\bar{X}).$$

So we have proved that

$$C_{\bar{P}}(\bar{X}) = \overline{C_P(S)} = \overline{C_P(S_{\bar{X}})} \tag{2}$$

We deduce that, if $C_{\bar{P}}(\bar{X}) \neq 1$, $C_P(S) \neq 1$. So $S \in \mathscr{X}$ and W_S is defined. Hence $W_{\bar{X}}$ is well-defined.

Now $P \in S_3(HP)$. So, if N is the pre-image in G of $N_{\bar{G}}(\bar{P})$, $HP \trianglelefteq N$ and, by the Frattini argument, $N = HPN(P) = HN(P)$. So

$$\bar{N} = N_{\bar{G}}(\bar{P}) = \overline{N(P)} \tag{3}$$

As $H \leqslant N(P)$, $H \leqslant PC(P)$. So, as $\overline{PC(P)} \leqslant \bar{P}C_{\bar{G}}(\bar{P})$, by (3), $N_{\bar{G}}(\bar{P})/\bar{P}C_{\bar{G}}(\bar{P})$ is a homomorphic image of $\overline{N(P)}/\overline{PC(P)} \cong N(P)/PC(P)$. By Hypothesis 4.1, $N_{\bar{G}}(\bar{P})/\bar{P}C_{\bar{G}}(\bar{P})$ is nilpotent. By (3), $N_{\bar{G}}(\bar{P})$ is solvable.

We note that, as $P \nleqslant H$, $3 \in \pi(\bar{G})$. We now verify (A1), (A2), (A3) and (A4) for $\bar{G}$. By (3), $\bar{U}$ is a Hall π-subgroup of $N_{\bar{G}}(\bar{P})$.

Fix $\bar{X} \leqslant \bar{U}$ such that $C_{\bar{P}}(\bar{X}) \neq 1$ and set $S = S_{\bar{X}}$. Then, by (2) and (A1),

$$W_{\bar{X}} = \overline{W}_S \leqslant \overline{\Omega_1(Z(C_P(S)))} \leqslant \Omega_1(Z(\overline{C_P(S)})) = \Omega_1(Z(C_{\bar{P}}(\bar{X}))).$$

We now have (A1) for $\bar{G}$.

Let $g \in G$ such that $(W_{\bar{X}})^g \leqslant \bar{P}$. Then $(W_S)^g \leqslant HP$. But $H \leqslant N(P)$. So $W_S{}^g \leqslant P$. By (A2), $g \in N(P)$. So, by (3), $\bar{g} \in N_{\bar{G}}(\bar{P})$. We have (A2) for $\bar{G}$.

Let $n \in N(P)$ such that $\bar{X}^{\bar{n}} \leqslant \bar{U}$. Then $S^n \leqslant HU$. As $H \leqslant N(P)$ and $N(P)$ is solvable, there is an $h \in H$ such that $S^{nh} \leqslant U$. We deduce that, by (A3) and (2), as $\bar{h} = 1$,

$$(W_{\bar{X}})^{\bar{n}} = (\overline{W}_S)^{\bar{n}} = \overline{(W_S)^n} = \overline{(W_S)^{nh}} = \overline{W_{S^{nh}}}.$$

Set $Z = S_{\bar{X}^{\bar{n}}}$. We want to show that $\overline{W_{S^{nh}}} = \overline{W_Z} = W_{\bar{X}^{\bar{n}}}$, as this would mean that (A3) holds for $\bar{G}$.

But $S^{nh} \leqslant U \cap V_{\bar{X}^{\bar{n}}} = Z$. Also $V_{\bar{X}^{\bar{n}}} = (V_{\bar{X}})^n$. Let $m = n^{-1}$. Then $Z^m \leqslant V_{\bar{X}}$ and Z^m is a π-group. As $H \leqslant N(P)$ and $\overline{V_{\bar{X}}} = \bar{X} \leqslant \bar{U}$, S is a Hall π-subgroup of $V_{\bar{X}}$. So

$$|Z| = |Z^m| \leqslant |S| = |S^{nh}|.$$

So, as $S^{nh} \leqslant Z$, $Z = S^{nh}$. But now $\overline{W_{S^{nh}}} = \overline{W_Z} = W_{\bar{X}^{\bar{n}}}$. So (A3) holds for $\bar{G}$.

Let $\bar{Y} \leqslant N_{\bar{U}}(\bar{X})$ and $u \in U$ such that $\bar{u} \in N_{\bar{U}}(\bar{X}) \cap N_{\bar{U}}(\bar{Y})$. Set $\bar{D} = [C_{W_{\bar{X}}}(\bar{Y}), \bar{u}]$ and $E = [C_{W_S}(S_{\bar{Y}}), u]$. We claim that $\bar{E} = \bar{D}$. Let $F = C_{W_S}(S_{\bar{Y}})$. Then

$$\bar{F} \leqslant \overline{C_{W_S}(S_{\bar{Y}})} \leqslant C_{\overline{W}_S}(\bar{S}_{\bar{Y}}) = C_{W_{\bar{X}}}(\bar{Y}). \tag{4}$$

So

$$\bar{E} = [\bar{F}, \bar{u}] \leqslant \bar{D}.$$

Let K be the pre-image in G of $C_{W_{\bar{X}}}(\bar{Y})$ and $T = P \cap K$. As $K \leqslant N(P)$, $T \in S_3(K)$ and $T \trianglelefteq K$. Also, by the same argument that established (2),

$T \leqslant C_P(S_{\bar{Y}})$. Furthermore $T \leqslant HW_S$. If $q \neq 3$ then, as $H \leqslant N(P)$, $W_S \in S_3(HW_S)$ and $W_S \trianglelefteq HW_S$. So $T \leqslant W_S$. If $q = 3$, $H \leqslant P$. So $H \leqslant Z(G)$. In any case

$$T \leqslant (Z(G) \cap P \cap H)W_S. \tag{5}$$

We deduce that, if $H_0 = Z(G) \cap P \cap H$, as $H_0 \leqslant C_P(S_{\bar{Y}})$, by (4),

$$T \leqslant C_P(S_{\bar{Y}}) \cap (H_0W_S) = H_0C_{W_S}(S_{\bar{Y}}) = H_0F \leqslant K \tag{6}$$

So $T = F$. As $\bar{E} \leqslant \bar{D}$ and $T \in S_3(K)$,

$$\bar{D} = [\bar{T}, \bar{u}] = [\bar{F}, \bar{u}] = \bar{E} \leqslant \bar{D}.$$

So $\bar{D} = [\bar{T}, \bar{u}] = \bar{E}$.

Now u normalises K. So, as $T = O_3(K)$, u normalises $T = F$. By Theorem 5.2.3 of [10], as $E = [F, u]$, $E \cap Z(G) = 1$. As E is a 3-group,

$$E \cap H = 1. \tag{7}$$

As u normalises $V_{\bar{X}}$ and $V_{\bar{Y}}$, u normalises S and $S_{\bar{Y}}$. We deduce that (A4) applies and so $E = 1$ or $m(E) \geqslant 3$. By (7), as $\bar{D} = \bar{E}$, either $\bar{D} = 1$ or $m(\bar{D}) \geqslant 3$. We have (A4) for $\bar{G}$. So Hypothesis 4.1 holds for $\bar{G}$ and we have (i).

(ii) is clear if $q \neq 3$. Suppose that $q = 3$. Let $\bar{B}$ be a $\bar{W}$-invariant $3'$-subgroup of $\bar{G}$ and B be the pre-image in G of $\bar{B}$. As $q = 3$, $H \leqslant O_3(G) \leqslant P$. So $H = H \cap P \leqslant Z(G)$. Then $B = H \times C$ where C is a $3'$-subgroup of B. But now, W normalises C.

If each W-invariant $3'$-subgroup of G lies in $C(W)$, then $C \leqslant C(W)$. So

$$\bar{B} = \bar{C} \leqslant \overline{C(W)} \leqslant C_{\bar{G}}(\bar{W}).$$

We deduce that (ii) holds. The lemma follows.

Lemma 4.10. *Assume Hypothesis* 4.1. *Suppose that* $C(O_3(G)) \leqslant N(P)$. *Then* $P \trianglelefteq G$.

Proof. Set $W = W_1$ (note $1 \in \mathscr{X}$). Let $n \in N(P)$. Then, by (A3), as $1^n = 1 \leqslant U$, $W = W_{1^n} = (W_1)^n = W^n$. So $W \trianglelefteq N(P)$. By (A2), W is weakly closed in P with respect to G. By (A2), $N(W) = N(P)$.

By (A1), $W \leqslant Z(P) \leqslant C_P(O_3(G))$. As $C(O_3(G)) \leqslant N(P)$, $C_P(O_3(G)) = O_3(C(O_3(G))) \leqslant G$. By 4.6(i), $G = N(P)$. So $P \trianglelefteq G$. The lemma is proved.

We note that the argument of 4.10 also proves

Corollary 4.11. *Assume Hypothesis* 4.1. *Let* $W = W_1$. *Then* W *is weakly closed in* P *with respect to* G, $W \leqslant \Omega_1(Z(P))$ *and* $N(W) = N(P)$.

LEMMA 4.12. *Assume Hypothesis* 4.1. *Let* $H \leqslant G$ *such that* $3 \in \pi(H)$ *and* $W_X \leqslant H$ *for each* $X \in \mathscr{X}$. *Suppose that* $U \cap H$ *is a Hall* π*-subgroup of* $N_H(P)$. *Set* $Q = P \cap H$. *Then* H *satisfies Hypothesis* 4.1 *with* P *replaced by* Q, U *replaced by* $U \cap H$ *and* W_X, π *unchanged. Also* $N_H(Q) = N_H(P)$.

Proof. Now $W \leqslant Q$. By 4.6(iii) and 4.11, $N(Q) \leqslant N(P)$. As $PC(P)$ is 3-nilpotent, $N_H(Q) \cap PC(P) \leqslant QC_H(Q)$. So $N_H(Q)/QC_H(Q)$ is a homomorphic image of $N_H(Q)/(N_H(Q) \cap PC_H(P)) \simeq N_H(Q)\, PC_H(P)/PC_H(P)$, a subgroup of a nilpotent group. So $N_H(Q)/QC_H(Q)$ is nilpotent.

As $N_H(Q) \leqslant N(P)$, $Q \in S_3(N_H(Q))$ and so $Q \in S_3(H)$. Also, as $N_H(Q) \leqslant N(P)$, $N_H(Q)$ is solvable. By assumption $3 \in \pi(H)$. We just need to verify (A1), (A2), (A3) and (A4) for H.

By assumption $U \cap H$ is a Hall π-subgroup of $N_H(P)$. But

$$N_H(P) \leqslant N_H(P \cap H) \leqslant N_H(Q) \leqslant N_H(P).$$

We conclude that

$$N_H(Q) = N_H(P). \tag{8}$$

Let $\mathscr{X}^* = \{X \leqslant U \cap H : C_Q(X) \neq 1\}$. As $Q \leqslant P$, $\mathscr{X}^* \subseteq \mathscr{X}$. By (8), $U \cap H$ is a Hall π-subgroup of $N_H(Q)$.

Clearly (A1) holds for H. Fix $X \in \mathscr{X}^*$. Let $h \in H$ such that $(W_X)^h \leqslant Q$. By (A2), as $Q \leqslant P$, $h \in N(P) \cap H$. By (8), $h \in N_H(Q)$. We have (A2).

But (A3) and (A4) for H are obvious. By (8), the lemma follows.

LEMMA 4.13. *Assume Hypothesis* 4.1. *Let* $X \in \mathscr{X}$ *and suppose that* $N_U(X)$ *is a Hall* π*-subgroup of* $N(X) \cap N(P)$.

For $Y \leqslant N_U(X)$ *such that* $C_P(X) \cap C_P(Y) \neq 1$, *let* $V_Y = W_{XY}$. *Then* $N(X)$ *satisfies Hypothesis* 4.1, *with* P *replaced by* $C_P(X)$, U *replaced by* $N_U(Y)$, W_Y *replaced by* V_Y *and* π *unchanged. Also* $N_{N(X)}(C_P(X)) = N_{N(X)}(P)$.

Proof. Set $N = N(X)$ and $Q = C_P(X)$. Now $[N_P(X), X] \leqslant P \cap X$. As $3 \notin \pi$, we conclude that

$$N_P(X) = Q. \tag{9}$$

But $W_X \leqslant Q$. So, by (A2), $N(Q) \leqslant N(P)$. By (9), $Q \in S_3(N_N(Q))$ and so

$$Q \in S_3(N) \qquad \text{and} \qquad N(Q) \leqslant N(P). \tag{10}$$

By (10) and Hypothesis 4.1, $N_N(Q)$ is solvable. As in the proof of 4.12, $N_N(Q)/QC_N(Q)$ is nilpotent. As $Q \neq 1$, $3 \in \pi(N)$. Clearly, by (9) and (10),

$$N_N(Q) = N_N(P). \tag{11}$$

So $N_U(X)$ is a Hall π-subgroup of $N_N(P)$. Let $H = N_U(X)$ and $Y \leqslant H$. Then

$$C_P(XY) = C_P(X) \cap C_P(Y) = C_Q(Y). \tag{12}$$

So, if $C_Q(Y) \neq 1$, $XY \in \mathscr{X}$ and W_{XY} is defined. So V_Y is well-defined.

Fix $Y \leqslant H$. By (A1) and (12),

$$V_Y \leqslant \Omega_1(Z(C_P(XY))) = \Omega_1(Z(C_Q(Y))).$$

We have (A1) for N.

Let $n \in N$ such that $(V_Y)^n \leqslant Q$. Then $(W_{XY})^n \leqslant P$. So, by (A2) and (11), $n \in N(P) \cap N = N_N(Q)$. We have (A2) for N.

Let $m \in N_N(Q)$ such that $Y^m \leqslant H$. Then $(XY)^m = XY^m \leqslant U$. By (A3),

$$(V_Y)^m = (W_{XY})^m = W_{XY^m} = V_{Y^m}.$$

We have (A3) for N.

Let $Z \leqslant N_H(Y)$ and $h \in N_H(Y) \cap N_H(Z)$. Then $[C_{V_Y}(H), Z] = [C_{W_{XY}}(Z), h]$. But $h \in N_U(XY) \cap N_U(Z)$. So (A4) holds for N. The lemma is proved.

LEMMA 4.14. *Assume Hypothesis* 4.1. *Let $X \in \mathscr{X}$. Then there is an $n \in N(P)$ such that $N_U(X^n)$ is a Hall π-subgroup of $N(P) \cap N(X^n)$ and $X^n \in \mathscr{X}$.*

Proof. Let H be a Hall π-subgroup of $N(P) \cap N(X)$. Then, as $N(P)$ is solvable, there is an $n \in N(P)$ such that $H^n \leqslant U$. As $(N(P) \cap N(X))^n = N(P) \cap N(X^n)$, it follows that $U \cap N(X^n)$ is a Hall π-subgroup of $N(P) \cap N(X^n)$. Also $X \leqslant O_\pi(N(X) \cap N(P)) \leqslant H$. So $X^n \leqslant H^n \leqslant U$. Also $C_P(X^n) = C_P(X)^n \neq 1$. Thus $X^n \in \mathscr{X}$. The lemma is proved.

LEMMA 4.15. *Assume Hypothesis* 4.1. *Let $W = W_1$ (note $1 \in \mathscr{X}$). Then we can suppose that the W_X satisfy the following properties.*

(i) *If $X \in \mathscr{X}$ then $W_X \geqslant C_W(X)$.*

(ii) *If $X \in \mathscr{X}$ such that $W \leqslant C_P(X)$ then $W_X = W$.*

Proof. For $X \in \mathscr{X}$, define V_X as follows. If $W \nleqslant C_P(X)$, let $V_X = W_X \cdot C_W(X)$. If $W \leqslant C_P(X)$, then let $V_X = W$.

We just have to verify (A1), (A2), (A3), (A4) for V_X. Fix $X \in \mathscr{X}$. By 4.11, $C_W(X) \leqslant \Omega_1(Z(P)) \cap C_P(X) \leqslant \Omega_1(Z(C_P(X)))$. We deduce that, by (A1), $V_X \leqslant \Omega_1(Z(C_P(X)))$. We have (A1) for V_X. By (A2), we have (A2) for V_X.

Let $n \in N(P)$ such that $X^n \leqslant U$. By 4.11, $W \trianglelefteq N(P)$. So $C_W(X^n) = C_W(X)^n$. By (A3), we may suppose that $W \nleqslant C_P(X)$. So

$$V_{X^n} = W_{X^n} \cdot C_W(X^n) = (W_X)^n \cdot (C_W(X))^n = (V_X)^n.$$

We have (A3) for V_X.

Let $Y \leqslant N_U(X)$ and $u \in N_U(X) \cap N_U(Y)$. By (A4), we may suppose $W \nleqslant C_P(X)$. By 2.3(ii),

$$C_{V_X}(Y) = C_{W_X}(Y) \cdot (C_W(X) \cap C(Y)) = C_{W_X}(Y) \cdot C_W(XY).$$

By (A4), we may suppose u does not centralise $C_{V_X}(Y)$. Also u normalises XY. By (A4), it easily follows that (A4) holds for V_X. The lemma is proved.

LEMMA 4.16. *Let a be an integer such that $a \geqslant 2$ and $G \cong PSL(2, 3^a)$. Let $\bar{\alpha}$ be an automorphism of G of order 2. Then $C_G(\bar{\alpha})$ does not contain a Sylow 3-subgroup of G.*

Proof. Suppose that $\bar{\alpha}$ centralises a Sylow 3-subgroup R of G. Now there is a group H such that $H \cong \Gamma L(2, 3^a)$ and $G \trianglelefteq H/Z(H)$. For $K \leqslant H$, let $\bar{K} = KZ(H)/Z(H)$. Then, as $H/Z(H)$ induces $\mathrm{Aut}(G)$ on G, there is a 2-element $t \in H$ such that $C_G(\bar{t})$ contains R and $t \notin Z(H)$.

Let $L \trianglelefteq H$ such that $L \cong SL(2, 3^a)$ and $Q \in S_3(L \cdot Z(H))$ such that $\bar{Q} = R$. Then $[\bar{Q}, \bar{t}] = [R, \bar{t}] = 1$. So $[Q, t] \leqslant Z(H)$. We deduce that t normalises $QZ(H)$, an abelian group. So t normalises Q. But $[Q, t, t] = 1$. By Theorem 5.3.6 of [10], $t \in C_H(Q)$.

Let V denote a two-dimensional vector space over $GF(3^a)$ on which H acts naturally, as a group of semilinear transformations. Let ϕ denote the Frobenius automorphism of $GF(3^a)$ and σ denote the semilinear transformation

$$\lambda x \mapsto \lambda^{\phi} x,$$

for $\lambda \in GF(3^a)$ and $x \in V$.

Let $M \trianglelefteq H$ such that $M \cong GL(2, 3^a)$. Then $H = M\langle\sigma\rangle$. With respect to some basis of V, we may suppose that $Q \geqslant \{\left[\begin{smallmatrix}1 & 0\\ \lambda & 1\end{smallmatrix}\right] : \lambda \in GF(3^a)\}$. Let $H_0 = \{\left[\begin{smallmatrix}b & 0\\ 0 & b\end{smallmatrix}\right] : b \in GF(3^a) - \{0\}\}$. Then $C_H(\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]) = \langle\sigma\rangle H_0 Q$. It follows that $C_H(Q) = QH_0$.

So $t \in H_0 \leqslant Z(H)$, in contradiction with the choice of t. So no such $\bar{\alpha}$ can exist and the lemma is proved.

LEMMA 4.17. *Assume Hypothesis* 4.1. *Assume further that $N(P) = PC(P)U$. Let $W = W_1$. Then*

(i) *If $U \leqslant C(P)$ then $G = O_{3'}(G) \cdot P$.*

(ii) *If $m(W) \leqslant 2$ then $U \leqslant C(W)$.*

(iii) *If P is cyclic then $G = O_{3'}(G) \cdot N(P)$.*

(iv) *Suppose that $m(W) \geqslant 3$, Let $q \in \pi(U)$ and $V \leqslant U$ such that $V \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Then there is an $x \in V^{\#}$ such that $m(C_W(x)) \geqslant 2$.*

Proof. Suppose that $U \leqslant C(P)$. Then

$$N(P) = PC(P) = P \times O_{3'}(N(P)).$$

By 4.11, $W \leqslant Z(P) \leqslant Z(J(P))$. By 4.11 and 4.6(i), $N(Z(J(P))) = N(P)$. So $N(Z(J(P)))$ is 3-nilpotent. By Theorem 8.3.1 of [10], G is 3-nilpotent. We have (i).

Suppose next that $m(W) \leqslant 2$. Let $u \in U$. By (A4), $m([W, u]) = 0$. So $U \leqslant C(W)$. We have (ii).

Suppose that P is cyclic. If $W = 1$ then, by (A2), $G = O_{3'}(G)\, N(P)$. If $W \neq 1$ then, by 4.11, $W = \Omega_1(P)$. But, by (ii), as $m(W) = 1$, U centralises W. So, by Theorem 5.2.4 of [10], $U \leqslant C(P)$. By (i), $G = O_{3'}(G) \cdot N(P)$. We have (iii).

Suppose that $q \in \pi(U)$ and $V \leqslant U$ such that $V \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Suppose further that $C_W(x)$ is cyclic for each $x \in V^{\#}$.

Let $x, y \in V^{\#}$. Then $[C_W(x), y] \leqslant C_W(x)$. As $C_W(x)$ is cyclic, by (A4), $C_W(x) \leqslant C_W(y)$. By symmetry of the argument we see that $C_W(x) = C_W(y)$ for each $x, y \in V^{\#}$.

Fix $x \in V^{\#}$. Then, by Theorem 6.2.4 of [10],

$$W = \langle C_W(y) : y \in V^{\#} \rangle = C_W(x).$$

As $m(W) \geqslant 3$, we have a contradiction. So, for some $z \in V^{\#}$, $m(C_W(z)) \geqslant 2$. We have (iv) and the lemma.

LEMMA 4.18. *Assume Hypothesis* 4.1. *Let* $W = W_1$. *Suppose that* $C(a) \leqslant N(P)$ *for each* $a \in W^{\#}$. *Let* $g \in G$ *such that* $W^g \cap P \neq 1$. *Then* $g \in N(P)$.

Proof. Let $h = g^{-1}$ and $Y = W \cap P^h$. Then $Y \neq 1$ and $C(Y) \leqslant N(P)$. Also $W^h \leqslant C(Y) \leqslant N(P)$. So $W^h \leqslant P$. By 4.11, $h \in N(P)$ and so $g \in N(P)$, which proves the lemma.

LEMMA 4.19. *Assume Hypothesis* 4.1. *Let* $W = W_1$, $q \in \pi(U)$ *and* $Q \in S_q(U)$. *Assume further that* $d(Q) \geqslant 2$ *and* $Q \notin S_q(G)$. *Also assume that the following conditions are satisfied.*

(a) $m(W) \geqslant 3$.

(b) *Whenever* $1 \neq X \leqslant Q$ *such that* $X \in \mathscr{X}$ *and* $m(C_W(X)) \geqslant 2$, $N(X) \leqslant N(P)$.

(c) *Whenever* $g \in G$ *such that* $W^g \cap P \neq 1$, $g \in N(P)$.

Then the following holds.

(i) $d(Q) = 2$.

(ii) $\Omega_1(Q) = \Omega_1(Z(Q)) \cong \mathbb{Z}_q \times \mathbb{Z}_q$.

(iii) *There exists an element* $x \in \Omega_1(Q)$ *such that* $C_P(x) = 1$.

Proof. As U is a Hall π-subgroup of $N(P)$, $Q \in S_q(N(P))$. So $Q \notin S_q(N(Q))$. We deduce that

(I) *There is a* $g \in N(Q) - Q$ *such that* $g \notin N(P)$ *but* $g^q \in Q$.

We also deduce from 4.17(iv) and (b) that

(II) *If* $V \leqslant Q$ *such that* $V \cong \mathbb{Z}_q \times \mathbb{Z}_q$, *then there is an* $x \in V^{\#}$ *such that* $m(C_W(x)) \geqslant 2$ *and* $C(x) \leqslant N(P)$. *So* $C(V) \leqslant N(P)$.

Suppose that $d(Q) \geqslant 3$ and let $V \leqslant Q$ such that $V \cong \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$. By (I) and 4.11, Q normalises W^g. So, by a well-known lemma,

$$W^g = \langle C_{W^g}(V_0) \colon m(V/V_0) \leqslant 1 \rangle.$$

It follows from (II) that $W^g \leqslant N(P)$. So $W^g \leqslant P$. By (A2), $g \in N(P)$, a contradiction. So $d(Q) < 3$. As $d(Q) \geqslant 2$, we have (i).

Let $V \leqslant Q$ such that $V \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Let $H = N_Q(V)$. Suppose that $H = C_Q(V)$. By (i), $V = \Omega_1(H) = \Omega_1(Z(H)) \trianglelefteq N_Q(H)$. So $N_Q(H) = H$ and $H = Q$. So, in order to prove (ii), we may suppose that $H \gneqq C_Q(V)$.

Let $K = \Omega_1(Z(H))$. As $d(Q) = 2$, $\Omega_1(Z(Q)) \leqslant K \leqslant V$. We deduce that, in order to prove (ii), we may suppose that

$$V \geqslant \Omega_1(Z(Q)) = K \cong \mathbb{Z}_q. \tag{13}$$

If $m(C_W(K)) \geqslant 2$, then, by (b), $N(K) \leqslant N(P)$. But now, by (13), $g \in N(P)$, a contradiction. We deduce that $m(C_W(K)) \leqslant 1$. By (a), $K \cap C(W) = 1$. As, by 4.11, $C_Q(W) \trianglelefteq Q$, and, by (13), $C_Q(W) \cap \Omega_1(Z(Q)) = 1$, we have that $C_Q(W) = 1$. So we have shown that we may suppose (13) holds, $m(C_W(K)) \leqslant 1$ and $C_Q(W) = 1$.

By (II), we can find $x \in V - K$ such that $m(C_W(x)) \geqslant 2$ and $C(x) \leqslant N(P)$. Let g be as in (I) and $h = g^{-1}$. By (c),

$$1 = W^g \cap P \geqslant C_W(x^h)^g \cap P.$$

But $C_W(x^h)^g \leqslant C(x) \leqslant N(P)$. So $C_W(x^h)^g \leqslant P$. We deduce that $C_W(x^h) = 1$.

Suppose that $q = 2$. As $C_Q(W) = 1$, H is represented faithfully on W. As $C_W(x^h) = 1$ and $o(x^h) = q = 2$, x^h inverts W. So $x^h \in K$. But, by (13), h normalises K. As $x \notin K$, we have a contradiction. So q is odd.

Set $R = Q \cdot \langle g \rangle$. We show that we may suppose $V \trianglelefteq R$. Set $L = \Omega_1(Z_2(Q))$. If $Z(L) \cong \mathbb{Z}_q \times \mathbb{Z}_q$, we choose $V = Z(L)$. Otherwise, as L has exponent q and class 2, and $d(Q) = 2$, L is extra-special of order q^3. We conclude that L has $(q+1)$ subgroups isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$. As $L \trianglelefteq R$, R permutes these subgroups and so R normalises one, which we can take to be V.

As $x \in V^{\#}$, $C_R(V) \leqslant Q$. As $V \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and $H \gneqq C_Q(V)$, $|R \colon C_R(V)| = q$. So, as $R \gneqq Q$,

$$q = |R \colon C_Q(V)| = |R \colon Q|.$$

So $Q = C_Q(V)$ and $H = C_Q(V)$, a contradiction. We conclude that (ii) holds.

Let $V = \Omega_1(Q) \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and $\mathscr{S}$ denote the set of subgroups of V of order q. Let g be as in (I) and $R = Q \cdot \langle g \rangle$. By (II), there is an $x \in V^\#$ such that $C(x) \leqslant N(P)$. So $C_R(x) = Q$.

Now g permutes the elements of $\mathscr{S}$. As $|\mathscr{S}| = q + 1$ and $C_R(\langle x \rangle) = Q$, there are two orbits of $\langle g \rangle$ on $\mathscr{S}$, viz. $\mathscr{O}$ and $\{\langle y \rangle\}$ where $|\mathscr{O}| = q$, $\langle x \rangle \in \mathscr{O}$ and $y \in V$.

Let $h \in \langle g \rangle$ such that $C_P(x^h) \neq 1$. Let $T = W_{\langle x^h \rangle}$ and $k = h^{-1}$. Then $T^k \leqslant C(x) \leqslant N(P)$. So $T^k \leqslant P$. By (A2), $k \in N(P)$. So $h \in N(P)$. We conclude from (I) that $h \in Q$. So $x^h = x$ and we have

(III) *If $z \in \mathscr{O} - \{\langle x \rangle\}$ then $C_P(z) = 1$.*

We have (iii) and the lemma.

COROLLARY 4.20. *Assume Hypothesis 4.1. Let $W = W_1$, $q \in \pi(U)$ and $Q \in S_q(U)$. Assume further that $d(Q) \geqslant 2$, $Q \notin S_q(G)$ and that the following conditions are satisfied.*

(a) *$m(W) \geqslant 3$.*

(b) *Whenever $1 \neq X \leqslant Q$ such that $d(C_P(X)) \geqslant 2$ then $N(X) \leqslant N(P)$.*

(c) *Whenever $g \in G$ such that $W^g \cap P \neq 1$ then $g \in N(P)$.*

Then the following holds.

(i) $d(Q) = 2$.

(ii) $\Omega_1(Q) = \Omega_1(Z(Q)) \cong \mathbb{Z}_q \times \mathbb{Z}_q$.

(iii) *There are elements $x, y \in Q$ such that $\Omega_1(Q) = \langle x, y \rangle$, $P \leqslant C(x)$ and $C_P(y) = 1$.*

Proof. Clearly (i) and (ii) follow from 4.19. We observe that 4.19 applies and continue from the proof of 4.19(iii).

Suppose $C_P(y) \neq 1$. Let $D = W_{\langle y \rangle}$. Let $B \in \mathscr{O} - \{\langle x \rangle\}$ and $b \in B^\#$. By (III), $C_P(b) = 1$. So, by (A4), $m(C_P(y)) \geqslant m([C_P(y), b]) \geqslant 3$. But, by (b), $C(y) \leqslant N(P)$. Also $g \in C(y)$. So $g \in N(P)$, a contradiction. We deduce that $C_P(y) = 1$. Now, by Theorem 6.2.4 of [10],

$$P = \langle C_P(B) : B \in \mathscr{S} \rangle.$$

We conclude from (III) that $P = C_P(\langle x \rangle) = C_P(x)$. As $\Omega_1(Q) = \langle x, y \rangle$, we have (iii) and the corollary.

LEMMA 4.21. *Let G be a group and q, r, t be primes such that $r \neq t$. Let H be a normal q-subgroup of G such that $H \cap O_t(G) \leqslant Z(G)$. For any subgroup K*

of G, let $\bar{K} = KH/H$. Let $R \in S_r(G)$ and S be a t-subgroup of $N(R)$. Suppose that $H \cap O_t(G) \leqslant S$ and $H \cap O_r(G) \leqslant Z(G)$. Then

(i) $N_{\bar{G}}(\bar{S}) = \overline{N(S)}$.

(ii) $C_{\bar{R}}(\bar{S}) = \overline{C_R(S)}$.

Proof. Let N be the pre-image in G of $N_{\bar{G}}(\bar{S})$. Then $HS \leqslant N$. If $t \notin \pi(H)$ then $S \in S_t(HS)$. Also, if $t \in \pi(H)$, $H = H \cap O_t(G) \leqslant S$. So $S = HS \in S_t(HS)$. By the Frattini argument,

$$N = HSN(S) = HN(S).$$

So $\bar{N} = N_{\bar{G}}(\bar{S}) = \overline{N(S)}$. We have (i).

Let T be the pre-image in G of $C_{\bar{R}}(\bar{S})$. Set $U = R \cap T$. If $q = r$ then $H \leqslant R$. We deduce that $U \in S_r(HR)$. Now $[\bar{U}, \bar{S}] \leqslant [C_{\bar{R}}(\bar{S}), \bar{S}] = 1$. So $[U, S] \leqslant H$. Now S normalises T and R. So S normalises U. Now $[U, S] \leqslant U \cap H \leqslant H \cap O_r(G) \leqslant Z(G)$. So $[U, S, S] = 1$. By Theorem 5.3.6 of [10], $[U, S] = 1$. So

$$C_{\bar{R}}(\bar{S}) = \bar{U} \leqslant \overline{C_R(S)} \leqslant C_{\bar{R}}(\bar{S}).$$

We deduce that $C_{\bar{R}}(\bar{S}) = \overline{C_R(S)}$. We have (ii) and the lemma.

LEMMA 4.22. *Assume Hypothesis* 4.1. *Suppose that q is a prime different from* 3 *such that $O_q(G) \cdot C(O_q(G)) \leqslant N(P)$. Then $P \lhd G$.*

Proof. Now $[O_q(G), P] \leqslant O_q(G) \cap P = 1$. So $P \leqslant C(O_q(G)) \leqslant N(P)$. But now $P = O_3(C(O_q(G))) \trianglelefteq G$, and the lemma follows.

We can now prove 4.2. We therefore introduce

HYPOTHESIS 4.23. G is a minimal counterexample to 4.2.

We show that $F(G) = 1$. We then argue that $SCN_3(2) = \phi$ and that P is abelian. We finally show that $E(G) \simeq L_2(3^a)$, $a \geqslant 2$, and that there is an $x \in C_T(P)^{\#}$ such that x induces an automorphism of $E(G)$ of order 2 which fixes a Sylow 3-subgroup of $E(G)$. We then use 4.16 to get a contradiction.

LEMMA 4.24. *Assume Hypothesis* 4.23. *Then $Z(G) = 1$ and $O_q(G) = 1$ for each $q \in \pi(G) - \{2, 3\}$.*

Proof. Let q be a prime. If $q \in \{2, 3\}$, let $H = O_q(Z(G))$. Otherwise let $H = O_q(G)$. For $K \leqslant G$, let $\bar{K} = KH/H$. As $P \ntriangleleft G$, $P \nleqslant H$. Choose W_X in accordance with 4.15 and define $W_{\bar{X}}$ as in 4.9.

Now $W_{\bar{1}} = \bar{W}$. So, by 4.9, $\bar{G}$, $\bar{P}$, $\bar{U}$, $W_{\bar{X}}$, π satisfy Hypothesis 4.1 and (B4). Also (B1) holds for $\bar{G}$. By 4.9(i) and (B2),

$$N_{\bar{G}}(\bar{P}) = \overline{N(P)} = \overline{PC(P)U} = \bar{P}C_{\bar{G}}(\bar{P})\bar{U}.$$

So (B2) holds for $\bar{G}$.

Let $T \in S_2(U)$. Then $\bar{T} \in S_2(\bar{U})$. By definition of H, $H \cap O_2(G) \leqslant O_2(N(P))$. By (B1), $T \in S_2(N(P))$. So $H \cap O_2(G) \leqslant C_T(P)$. By 4.22, (B3) and 4.21(i),

$$N_{\bar{G}}(C_{\bar{T}}(\bar{P})) = N_{\bar{G}}(\overline{C_T(P)}) \leqslant \overline{N(P)} = N_{\bar{G}}(\bar{P}).$$

Hence (B3) holds for $\bar{G}$. So $\bar{G}$ satisfies the hypotheses of 4.2. If $\bar{P} \lhd \bar{G}$ then $HP \lhd G$. By (B4), if $q \neq 3$, $H \leqslant C(W)$. By 4.11, $H \leqslant N(P)$, and so $P \lhd HP$. But, if $q = 3$, $H \leqslant P$. So $P = O_3(HP) \lhd G$, a contradiction. By minimality of G, $H = 1$. The lemma is proved.

LEMMA 4.25. *Assume Hypothesis* 4.23. *Let* $W_0 = \langle W_X : X \in \mathscr{X} \rangle$ *and* $W^* = \langle W_0^{N(P)} \rangle$. *Let Z be a 2-subgroup of $C(P)$ such that $C(Z) \leqslant N(P)$. Let $H \lneqq G$ such that $W^*Z \leqslant H$. Then $H \leqslant N(P)$.*

Proof. Observe that $W^* \leqslant P$. Let $Q \in S_3(H)$ such that $Q \geqslant W^*$. By 4.11 and 4.6(i), $N(Q) \leqslant N(P)$. In particular $Q = P \cap H$. By (B1), there is a Hall π-subgroup V of $N_H(P)$ such that $V \geqslant Z$. As $N(P)$ is solvable, there is an $n \in N(P)$ such that $V^n \leqslant U$. Let $K = H^n$. It follows that $L = U \cap N_K(P)$ is a Hall π-subgroup of $N_K(P)$. Also $Z^n \leqslant L$ and $C(Z^n) = C(Z)^n \leqslant N(P)$. Also $W^* = W^{*n} \leqslant K$. So we may suppose $H = K$, $Z^n = Z$.

Let $S \in S_2(L)$. Then there is an $m \in L$ such that $Z^m \leqslant S$. It follows that $Z^m \leqslant C(P) \cap S \leqslant C_S(Q)$. But now

$$C(C_S(R)) \leqslant C(Z^m) = C(Z)^m \leqslant N(P).$$

So $C_H(C_S(Q)) \leqslant N(P) \cap H \leqslant N_H(Q) \leqslant N(P)$. As $Q \leqslant C_H(Q))$, by the Frattini argument,

$$N_H(C_S(Q)) = C_H(C_S(Q)) \cdot (N_H(Q) \cap N_H(C_S(Q)) \leqslant N_H(P) \leqslant N_H(Q).$$

By 4.12, H satisfies the hypotheses of 4.2. As $H \lneqq G$, $Q \trianglelefteq H$ and so $H \leqslant N(P)$. The lemma is proved.

Fix $T \in S_2(U)$ and set $Z = \Omega_1(Z(T)) \cap C_T(P)$. By (B1) and the definition of U, $T \in S_2(N(P))$.

COROLLARY 4.26. *Assume Hypothesis* 4.23. *Then*

(i) $N(Z) \leqslant N(P)$.

(ii) *Let* $a \in Z(P)^{\#}$. *Then* $C(a) \leqslant N(P)$.

(iii) $F(G) = 1$.

Proof. By (B3), as $P \ntrianglelefteq G$, $Z \neq 1$. By 4.24, $C(Z) \lneqq G$. But $PC_T(P) \leqslant C(Z)$. So, by 4.25, $C(Z) \leqslant N(P)$. By the Frattini argument, as $P \leqslant C(Z)$, $N(Z) = C(Z) \cdot (N(Z) \cap N(P)) \leqslant N(P)$. We have (i).

Let $a \in Z(P)^{\#}$. Then $PC_T(P) \leqslant C(a)$. By 4.24, $C(a) \lneqq G$. So, by 4.25, $C(a) \leqslant N(P)$. We have (ii).

Suppose that $O_3(G) \neq 1$. Then $Z(P) \cap O_3(G) \neq 1$. By (ii), $C(O_3(G)) \leqslant N(P)$. But now, by 4.10, $P \trianglelefteq G$, a contradiction. So $O_3(G) = 1$.

Now suppose that $F(G) \neq 1$. Let $q \in \pi(F(G))$. By 4.24, $q \in \{2, 3\}$. By 4.11 and (B4), $O_q(G) \leqslant N(P)$. As $T \in S_2(N(P))$, $O_2(G) \leqslant T$. We deduce that $PZ \leqslant C(O_q(G)$. By 4.24, 4.25 and (i), $C(O_q(G)) \leqslant N(P)$. By 4.22, $P \trianglelefteq G$, which is a contradiction. So $F(G) = 1$. We have (iii) and the corollary.

LEMMA 4.27. *Assume Hypothesis* 4.23. *Then*

(i) $W \nleqslant C_P(U)$.

(ii) $m(W) \geqslant 3$.

Proof. By (A4), it is enough to prove (i). Suppose $W \leqslant C_P(U)$. By 4.15, we can suppose $W_X = W$ for each $X \in \mathscr{X}$. Let $1 \neq V \leqslant T$. Then, using the notation of 4.25, $W^* = W \leqslant N(V)$. Also $Z \leqslant N(V)$. By 4.26(iii) and 4.25, $N(V) \leqslant N(P)$. In particular, as $T \in S_2(N(P))$, $T \in S_2(N(T))$ and so $T \in S_2(G)$.

Let N be the subgroup of G generated by all the involutions of G. By 4.26(iii), as $N \trianglelefteq G$, N is not solvable. So, as $N(P)$ is solvable, $N \nleqslant N(P)$ and so not all involutions of G lie in $N(P)$. We conclude that $N(P)$ is a strongly embedded subgroup of G.

By the main theorem of [5], $O(G)$ is solvable. So, by 4.26(iii), $O(G) = 1$. By the corollary on page 89 of [22], $O(N(P)) = 1$, a contradiction as $P \ntrianglelefteq G$. Thus $W \nleqslant C_P(U)$ and the lemma is proved.

LEMMA 4.28. *Assume Hypothesis* 4.23. *Let* $1 \neq X$ *be a* 2-*subgroup of* U *such that* $X \in \mathscr{X}$. *Assume further that each* W_X-*invariant* 3'-*subgroup of* G *lies in* $C(W_X)$. *Then* $N(X) \leqslant N(P)$.

Proof. By 4.14 there is an $n \in N(P)$ such that $X^n \in \mathscr{X}$ and $N_U(X^n)$ is a Hall π-subgroup of $N(X^n) \cap N(P)$. By (A3), $W_{X^n} = (W_X)^n$. It follows that we may suppose $X \leqslant T$ and that $N_U(X)$ is a Hall π-subgroup of $N(P) \cap N(X)$.

Set $N = N(X)$, $D = N_U(X)$, $Q = C_P(X)$ and adopt the notation of 4.13. By 4.13, N, Q, V_Y, π satisfy Hypothesis 4.1 and

$$N_N(Q) = N_N(P). \tag{15}$$

By (B2), $O^\pi(N_N(P)) \leqslant O^\pi(N(P)) \leqslant PC(P)$. As $PC(P)$ is 3-nilpotent and $Q \in S_3(N)$, $O^\pi(N_N(P)) \leqslant QC_N(Q)$. As D is a Hall π-subgroup of $N_N(P)$, we have (B2) for N.

Now $V_1 = W_X$. So, by (B1) and the hypothesis of the lemma, (B1) and (B4) hold for N. We now check (B3) for N.

Now $Z \leqslant C(P) \cap U \cap N \leqslant C_D(Q)$. So, for $S \in S_2(D)$, there is a $d \in D$ such that $Z^d \leqslant S \cap C(Q) = C_S(Q)$. So, by 4.26(i) and (15),

$$C_N(C_S(Q)) \leqslant C(Z^d) \cap N = C(Z)^d \cap N \leqslant N_N(P) = N_N(Q).$$

So, by the Frattini argument, as $Q \in S_3(C_N(C_S(Q)))$, $N_N(C_S(Q)) \leqslant N_N(Q)$. So N satisfies the hypotheses of 4.2. By 4.26(iii), $N < G$. So, by minimality of G, and (15), $N = N_N(Q) \leqslant N(P)$. The lemma is proved.

Corollary 4.29. *Assume Hypothesis* 4.23. *Let* $1 \neq X \leqslant T$ *such that* $m(C_W(X)) \geqslant 2$. *Then* $N(X) \leqslant N(P)$.

Proof. Choose W_X in accordance with 4.15. Then $W_X \geqslant C_W(X)$. Let B be a W_X-invariant $3'$-subgroup of G. Then, by Theorem 6.2.4 of [10],

$$B = \langle C_B(a) \colon a \in C_W(X)^{\#} \rangle.$$

By 4.11 and 4.26(ii), $B \leqslant N(P)$. So $[B, W_X] \leqslant B \cap P = 1$. By 4.28, the corollary follows.

Lemma 4.30. *Assume Hypothesis* 4.23. *Suppose that* $T \notin S_2(G)$. *Then*

(i) P *is abelian.*

(ii) $C(a) \leqslant N(P)$ *for each* $a \in P^{\#}$.

(iii) $SCN_3(2) = \emptyset$.

Proof. As $P \ntrianglelefteq G$, by (B3), $C_T(P) \neq 1$. If $d(T) = 1$ then $\Omega_1(T) \leqslant C_T(P)$. But, by 4.27(ii) and 4.29, $N(T) \leqslant N(\Omega_1(T)) \leqslant N(P)$. As $T \in S_2(N(P))$, $T \in S_2(G)$, a contradiction. So $d(T) \geqslant 2$.

By 4.27(ii), 4.29 and 4.18, 4.19 applies to T. By 4.19(iii), there is an involution $t \in T$ such that $C_P(t) = 1$. So P is abelian. We have (i). By 4.26(ii), we have (ii).

By 4.19(i), $d(T) = 2$. Let t be an involution of Z. By 4.25 and 4.26(i), as $PZ \leqslant C(t)$, $C(t) \leqslant N(P)$. Let $S \in S_2(G)$, $S \geqslant T$. Let $A \in SCN_3(S)$. By considering the Jordan canonical form of t acting on $\Omega_1(A)$, we see that $m(C_A(t)) \geqslant 2$. As $C(t) \leqslant N(P)$, $C_S(t) = T$. So, as $d(T) = 2$, $t \in C_A(t)$. But now $A \leqslant C_S(t) = T$, a contradiction. Thus $SCN_3(S) = \emptyset$ and we have (iii) and the lemma.

Lemma 4.31. *Assume Hypothesis* 4.23. *Assume further that* $T \in S_2(G)$. *Then there is a* $g \in G - N(P)$ *such that* $Z^g \leqslant T$.

Proof. Suppose that whenever $g \in G$ such that $Z^g \leqslant T$, then $g \in N(P)$. As $T \in S_2(N(P))$, we have

(I) *Let* $g \in G$ *such that* $Z^g \leqslant N(P)$. *Then* $g \in N(P)$.

Set $K = O^2(G) \cdot C_T(P)$ and $S = T \cap K$. Then $S \in S_2(K)$.

Let $t \in S$ and $g \in K$ such that $t^g \in S$. Then $Z, Z^g \leqslant C_K(t^g)$. Let $V \in S_2(C_K(t^g))$, $V \geqslant Z$. By (I), as $Z \leqslant V \cap N(P)$, $N_V(V \cap N(P)) \leqslant V \cap N(P)$. So $V = V \cap N(P) \leqslant N(P)$. Let $c \in C_K(t^g)$ such that $Z^{gc} \leqslant V$. Then, by (I), $gc \in N(P)$. As $t^g = t^{gc}$, we have proved,

(II) *Let $g \in K$ and $t \in S$ such that $t^g \in S$. Then there is an $n \in N_K(P)$ such that $t^g = t^n$.*

Now $C_T(P) \trianglelefteq S$. Let D be the pre-image in S of $(S/C_T(P))'$. Let θ denote the transfer homomorphism $K \to S/D$. Using the notation of Theorem 7.3.3 of [10],

$$x\theta = \prod_{i=1}^{n} (x_i x^{r_i} x_i^{-1})\phi, \tag{16}$$

where ϕ denotes the natural map $S \to S/D$.

As in 4.13, $N_K(P)/PC_K(P)$ is nilpotent. For $X \leqslant N_K(P)$, let $\bar{X} = XPC_K(P)/PC_K(P)$. Let $n \in N_K(P)$ and $t \in T$ such that $t^n \in T$. Then $\bar{T} \in S_2(\overline{N_K(P)})$. As $\overline{N_K(P)}$ is nilpotent, $\bar{t}^{\bar{n}} = \bar{t}^{\bar{s}}$, some $s \in T$. So

$$\bar{t}^{-1}\bar{t}^{\bar{n}} = \bar{t}^{-1}\bar{t}^{\bar{s}}.$$

So, for some $a \in PC_K(P)$, $t^{-1}t^n = t^{-1}t^s a$. As $t^n \in T$, $a \in C_T(P)$. So $t^{-1}t^n \in D$. we deduce that

(III) *If $t \in T$ and $n \in N_K(P)$ such that $t^n \in T$, then $tD = t^n D$.*

By (II), (III) and (16), we see that if $t \in S$,

$$t\theta = (t\phi)^{|K:S|}.$$

As $S \in S_2(K)$, $t\theta \neq 1$ if and only if $t\phi \neq 1$. But now $C_T(P) \leqslant \ker \theta$. So, if $H = O^2(G)$,

$$\ker \theta = C_T(P) \cdot (\ker \theta \cap H).$$

But $H/(\ker \theta \cap H) \cong H \ker \theta/\ker \theta$, a 2-group. As $H = O^2(H)$, $\ker \theta = K$. So $S = D$ and $(S/C_T(P)) = (S/C_T(P))'$. We conclude that $S = C_T(P)$.

By 4.25 and 4.26, $C(t) \leqslant N(P)$ for each $t \in C_T(P)^{\#}$. Let $S^* = C_T(P) \cap H$. Then $S^* \in S_2(H)$ and, for each $t \in S^{*\#}$, $C_H(t) \leqslant N_H(P)$.

Suppose that $S^* \neq 1$. Then $P \leqslant C_H(S^*) \leqslant N(P)$. By the Frattini argument, $N_H(S^*) \leqslant N_H(P)$. It follows as in the proof of 4.27(i) that $O(G) = 1$ and $N_H(P)$ is a strongly embedded subgroup of G. As $H \trianglelefteq G$, $O(H) = 1$. But now, by the corollary on p. 89 of [22], $O(N(P)) = 1$, a contradiction. So $S^* = 1$. But now $H = O(H) \leqslant O(G) = 1$ and G is a 2-group, also a contradiction. The lemma follows.

LEMMA 4.32. *Assume Hypothesis* 4.23. *Then*

(i) *P is abelian.*

(ii) *$C(a) \leqslant N(P)$ for each $a \in P^{\#}$.*

(iii) $SCN_3(2) = \emptyset$.

Proof. By 4.30, we may suppose $T \in S_2(G)$. By 4.31, there is a $g \in G$ such that $g \notin N(P)$ but $Z^g \leqslant T$. Let $\sigma \in Z^{\#}$. By 4.25 and 4.26, $C(\sigma) \leqslant N(P)$, so that $P = O_3(C(\sigma))$. So $P^g = O_3(C(\sigma^g))$.

Suppose that $C_P(\sigma^g) \neq 1$. Then $\langle \sigma^g \rangle \in \chi$ and so we can set $D = W_{\langle \sigma^g \rangle}$. As $D \leqslant C(\sigma^g)$, $D \leqslant P^g$. But now, by (A2), $g \in N(P)$, a contradiction. So $C_P(\sigma^g) = 1$. As $o(\sigma^g) = 2$, P is abelian. We have (i). By 4.26(ii), we have (ii).

Suppose that $d(C_T(\sigma^g)) \geqslant 3$. Let $V \leqslant C_T(\sigma^g)$ such that V is elementary abelian of order 8. By a well known lemma,

$$P^g = \langle C_{P^g}(V_0) \colon m(V_0) = 2 \rangle. \tag{17}$$

Let $V_0 \leqslant V$ such that $m(V_0) = 2$. By 4.17(iv), 4.27, and 4.29, there is an $x \in V_0^{\#}$ such that $m(C_W(x)) \geqslant 2$ and $C(x) \leqslant N(P)$. Thus $C(V_0) \leqslant N(P)$. It follows from (17) that $P^g \leqslant N(P)$. So $P^g = P$ and $g \in N(P)$, a contradiction. We deduce that $d(C_T(\sigma^g)) \leqslant 2$.

Using the proof of 4.30(iii), we have that $SCN_3(T) = \emptyset$. As $T \in S_2(G)$, we have (iii) and the lemma.

LEMMA 4.33. *Assume Hypothesis* 4.23. *Then*

(i) *$E(G)$ is simple.*

(ii) $m(P \cap E(G)) \geqslant 3$.

Proof. Let $E = E(G)$. Suppose that $3 \notin \pi(E)$. By 4.11 and (B4), $E \leqslant N(P)$. But, by Hypothesis 4.1, $N(P)$ is solvable. So $E = 1$. But now, by 4.26(iii), $F^*(G) = 1$. So $G = 1$, a contradiction.

Thus $3 \in \pi(E)$. Let $Q = P \cap E$. Then $Q \in S_3(E)$. Let E_1 be a component of E such that $3 \in \pi(E_1)$. Then $Q \cap E_1 \neq 1$. By 4.32(ii), as $N(P)$ is solvable, $C(E_1)$ is solvable. It follows that $E_1 = E$. By 4.26(iii), E is simple and we have (i).

By 4.32(i), Q is abelian. Suppose that $m(Q) \leqslant 2$. By 4.32, $P \leqslant C(Q) \leqslant N(P)$. So $P = O_3(C(Q)) \trianglelefteq N(Q)$ and $N(Q) \leqslant N(P)$. By (i), E is not 3-nilpotent. So, by Theorem 7.4.3 of [10], $N_E(Q) > C_E(Q)$. As Q is abelian and $m(Q) \leqslant 2$, $| N_E(Q) : C_E(Q)|$ is even. Thus $| N_E(Q)|$ is even.

Set $S = T \cap N_E(Q)$. Now $N_E(Q) = N_E(P \cap E) \trianglelefteq N(P)$. So $S \in S_2(N_E(Q))$ and so $S \neq 1$. Let $t \in S$. Then $[W, t] \leqslant P \cap E = Q$. So $m([W, t]) \leqslant 2$. By (A4), $[W, t] = 1$. So

$$W \leqslant C(S). \tag{18}$$

Let $1 \neq X \leqslant S$. Then $W = C_W(X)$. By 4.27(ii), $m(C_W(X)) \geqslant 2$. So, by 4.29, $N(X) \leqslant N(P)$. So

$$N_E(X) = N(P) \cap E \leqslant N_E(Q).$$

We conclude that $N_E(Q)$ is a strongly embedded subgroup of E. But now, by the corollary on p. 89 of [22], $Q \leqslant O(N_E(Q)) = 1$, a contradiction. So $m(Q) \geqslant 3$ and we have (ii) and the lemma.

LEMMA 4.34. *Assume Hypothesis* 4.23. *Let* $E = E(G)$ *and* $Q = P \cap E$. *Then*

(i) $E \cong PSL(2, 3^a)$, $a \geqslant 2$.

(ii) *Let t be an involution of $C_T(P)$. Then t induces an automorphism of E of order* 2 *such that* $Q \leqslant C_E(t)$.

Proof. By 4.32, $SCN_3(2) = \emptyset$. By a result of MacWilliams (See introduction of [11]), a Sylow 2-subgroup of G has sectional 2-rank most 4. By a result of Gorenstein and Harada [11] and a list of known simple groups with a strongly 3-embedded subgroup compiled by McBride [20], $m(Q) \leqslant 2$ or $E \cong L_2(3^a)$, $U_3(3^a)$ or E is of Ree type. By 4.33(ii) and 4.32(i), Q is abelian and $m(Q) \geqslant 3$. So $E \cong L_2(3^a)$, $a \geqslant 2$. We have (i).

Let t be an involution of $C_T(P)$. By 4.26(iii), $F^*(G) = E$. So, as $C(F^*(G)) \leqslant F^*(G)$, t induces an automorphism of E of order 2 by conjugation. As $P \leqslant C(t)$, $Q \leqslant C_E(t)$. We have (ii) and the lemma.

Proof of 4.2. Now 4.34 and 4.16 contradict each other. So no minimal counterexample to 4.2 can exist and so 4.2 is proved.

Wr now prove 4.3. We therefore introduce the following hypothesis.

HYPOTHESIS 4.35. G is a minimal counterexample to 4.3.

We use an argument reminiscent of [6]. Noting that $1 \in \mathscr{X}$, set $W = W_1$.

LEMMA 4.36. *Assume Hypothesis* 4.35. *Then*

(i) $d(P) \geqslant 2$.

(ii) $Z(G) = O_3(G) = 1$.

Proof. (i) is immediate from 4.17(iii).

Suppose $O_3(G) \neq 1$. As $O_3(G) \leqslant P$, by (C3), $C(O_3(G)) \leqslant N(P)$. By 4.10, $P \trianglelefteq G$, a contradiction. So $O_3(G) = 1$.

Suppose that $Z(G) \neq 1$. Let $q \in \pi(Z(G))$ and $H = O_q(Z(G))$. If $K \leqslant G$, let $\bar{K} = KH/H$. Now $H \leqslant N(P)$ and $P \nleqslant H$. Let $W_{\bar{X}}$ be as in 4.9. By 4.9(i), $\bar{G}$, $\bar{P}$, $\bar{U}$, $W_{\bar{X}}$ and π satisfy Hypothesis 4.1. Also $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$. Clearly (C1) holds for $\bar{G}$. By 4.9(i) and (C2),

$$N_{\bar{G}}(\bar{P}) = \overline{N(P)} = \overline{PC(P)U} = \bar{P}C_{\bar{G}}(\bar{P})\bar{U}.$$

We have (C2) for $\bar{G}$.

Let $a \in P^{\#}$. By (C3), $C(H \cdot \langle a \rangle) \leqslant N(P)$. So $W \leqslant C_P(H \cdot \langle a \rangle) = O_3(C(H \cdot \langle a \rangle)) \trianglelefteq N(H \cdot \langle a \rangle)$. By 4.11 and 4.6(i), $N(H \cdot \langle a \rangle) \leqslant N(P)$. But $H \cdot \langle a \rangle \trianglelefteq C_G(H \cdot \langle a \rangle \text{ mod. } H)$. So $C_{\bar{G}}(\bar{a}) \leqslant N_{\bar{G}}(\bar{P})$. We have (C3) for $\bar{G}$.

Let $\bar{V}$ be a r-subgroup of $\bar{U}$. Then $V \leqslant HU$. If $q = r$, $H \leqslant O_\pi(N(P)) \leqslant U$. It follows that if $V_r \in S_r(U \cap V)$ then $V_r \in S_r(V)$. By 4.21(i), $N_{\bar{G}}(\bar{V}_r) = \overline{N(V_r)}$. So, by (C4),

$$O^r(N_{\bar{G}}(\bar{V}_r)) = \overline{O^r(N(V_r))} \leqslant \overline{C(V_r)} \leqslant C_{\bar{G}}(\bar{V}_r).$$

We have (C4) for $\bar{G}$.

So, as $H \neq 1$, by minimality of G, $\bar{G} = O_{3'}(\bar{G}) \cdot N_{\bar{G}}(\bar{P})$. As $q \neq 3$, $H \leqslant O_{3'}(G)$. Also $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$. So $\bar{G} = \overline{O_{3'}(G) \cdot N(P)}$. But now $G = HO_{3'}(G)\, N(P) = O_{3'}(G)\, N(P)$, a contradiction. So $Z(G) = 1$. The lemma follows.

LEMMA 4.37. *Assume Hypothesis* 4.35. *Then*

(i) *U is nilpotent.*

(ii) *Let $H \lneqq G$ such that $H \cap U$ is a Hall π-subgroup of $N_H(P)$ and $W_X \leqslant H$ for each $X \in \mathscr{X}$. Suppose that $d(P \cap H) \geqslant 2$. Then $H \leqslant N(P)$.*

(iii) $F(G) = 1$.

(iv) *$G = O^q(G)$ for each $q \in \pi(G) - \{3\}$.*

(v) *Let $q \in \pi(U)$ and $V \leqslant U$ such that $V \simeq \mathbb{Z}_q \times \mathbb{Z}_q$. Then there is an $x \in V^{\#}$ such that $d(C_P(x)) \geqslant 2$.*

Proof. By (C4) and Theorem 7.4.5 (a) of [10], U is q-nilpotent for each $q \in \pi(U)$. So U is nilpotent and we have (i).

Let $H \lneqq G$ such that $U \cap H$ is a Hall π-subgroup of $N_H(P)$, $W_X \leqslant H$ for each $X \in \mathscr{X}$ and $d(P \cap H) \geqslant 2$. Set $Q = P \cap H$. By 4.12, H, Q, $U \cap H$, W_X and π satisfy Hypothesis 4.1. Also $N_H(P) = N_H(Q)$.

Clearly (C1) holds for H. As in 4.12, $N_H(Q)/QC_H(Q)$ is isomorphic to a section of $N_H(P)/PC_H(P)$. By (C2), $N_H(P)/PC_H(P)$ is a π-group. So $N_H(Q)/QC_H(Q)$ is a π-group. As $U \cap H$ is a Hall π-subgroup of $N_H(P) = N_H(Q)$, (C2) holds for H.

But, as $N_H(P) = N_H(Q)$, (C3) and (C4) hold for H. By minimality of G, $H = O_{3'}(H) \cdot N_H(Q)$.

By Theorem 6.2.4 of [10], and (C3), as $d(Q) \geqslant 2$, $O_{3'}(H) \leqslant N(P)$. As $N_H(Q) = N_H(P)$, $H \leqslant N(P)$. We have (ii).

By 4.36(i), (C3) and Theorem 6.2.4 of [10], $O_{3'}(G) \leqslant N(P)$. Let $q \in \pi(G) - \{3\}$ and suppose that $O_q(G) \neq 1$. Then $O_q(G) \leqslant N(P)$. Set $C = C(O_q(G))$. Then $P \leqslant C$ and $N_C(P) \trianglelefteq N(P)$. So $U \cap C$ is a Hall π-subgroup of $N_C(P)$. By 4.36, as $Z(G) = 1$ and $d(P) \geqslant 2$, $C \lneqq G$ and so, by (ii), $C \leqslant N(P)$. But now, by 4.22, $P \trianglelefteq G$, a contradiction. We deduce that $O_{3'}(F(G)) = 1$. So, by 4.36(ii), $F(G) = 1$. We have (iii).

Let $q \in \pi(G) - \{3\}$ and $H = O^q(G)$. Then $P \leqslant H$. If $H \leqslant N(P)$ then $P = O_3(H) \trianglelefteq G$, a contradiction. So $H \nleqslant N(P)$. As $N_H(P) \trianglelefteq N(P)$, $U \cap H$ is a Hall π-subgroup of $N_H(P)$. By 4.36(i), $d(P) \geqslant 2$. So, by (ii), $H = G$. We have (iv).

Let $q \in \pi(U)$ and $V \leqslant U$ such that $V \simeq \mathbb{Z}_q \times \mathbb{Z}_q$. Suppose that $C_P(x)$ is cyclic for each $x \in V^{\#}$. Let $x, y \in V^{\#}$. We claim that $C_P(x) = C_P(y)$.

Suppose not and let $x, y \in V^{\#}$ such that $C_P(x) \neq C_P(y)$. We may suppose that $| C_P(x)| \geqslant | C_P(y)|$. As V is abelian, y normalises $C_P(x)$. Also $C_P(x) \neq 1$, so that $\langle x \rangle \in \mathscr{X}$. Let $D = W_{\langle x \rangle}$. Then $[D, y] \leqslant C_P(x)$, a cyclic group. By (A4), $[D, y] = 1$. By (A2), as G is a counterexample, $D \neq 1$. So $D = \Omega_1(C_P(x))$. By Theorem 5.2.4 of [10], $C_P(x) \leqslant C_P(y)$, a contradiction. The claim is justified.

Fix $x \in V^{\#}$. By Theorem 6.2.4 of [10], as $m(V) = 2$, $P = C_P(x)$. But, by 4.36(i), $d(C_P(x)) \geqslant 2$, a contradiction. We deduce that (v) holds. The lemma is proved.

LEMMA 4.38. *Assume Hypothesis* 4.35. *Then the following holds.*

(i) *Let $r \in \pi(U)$ and $R \in S_r(U)$. Then $R \notin S_r(G)$.*

(iii) *Let $1 \neq X \leqslant U$ such that $d(C_P(X)) \geqslant 2$. Then $N(X) \leqslant N(P)$.*

Proof. Let $r \in \pi(U)$ and $R \in S_r(U)$. If $R \in S_r(G)$ then, by (C4), if S is an r-subgroup of G, $N(S)/C(S)$ is an r-group. By Theorem 7.4.5 (a) of [10], $O^r(G) \lneqq G$. As $3 \notin \pi$, this contradicts 4.37(iv). So $R \notin S_r(G)$. We have (i).

Let $1 \neq X \leqslant U$ such that $d(C_P(X)) \geqslant 2$. Let $N = N(X)$. By 4.14, we may suppose that $U \cap N$ is a Hall π-subgroup of $N_N(P)$. Adopt the notation of 4.13. Let $Q = C_P(X)$.

Then N, Q, $U \cap N$, V_Y, π satisfy Hypothesis 4.1. Also $N_N(P) = N_N(Q)$. So N satisfies (C1), (C3) and (C4). But, using the argument of the proof of 4.37 (ii), (C2) also holds for N. So N satisfies the hypotheses of 4.3.

By 4.37(iii), $N \lneqq G$. So, by minimality of G, $N = O_{3'}(N) \cdot N_N(Q)$. But, by (C3) and Theorem 6.24 of [10], as $d(Q) = d(C_P(X)) \geqslant 2$, $O_{3'}(N) \leqslant N_N(P)$. So, as $N_N(Q) = N_N(P)$, $N \leqslant N(P)$. We have (iii) and the lemma.

LEMMA 4.39. *Assume Hypothesis* 4.35. *Assume further that $C_P(U) \neq 1$. Then the following holds.*

(i) *Let $r \in \pi(U)$ and $R \in S_r(U)$. Then $C(R) \nleqslant N(P)$ and $d(R) = 1$.*

(ii) *$C_P(U)$ is cyclic.*

(iii) *Let $1 \neq X \leqslant U$. Then $C_P(X) = C_P(U)$.*

Proof. As $C_P(U) \neq 1$, $U \in \mathscr{X}$. Let $D = W_U$. Let $r \in \pi(U)$ and $R \in S_r(U)$. Suppose that $C(R) \leqslant N(P)$. Then $D \leqslant C_P(R) = O_3(C(R)) \trianglelefteq N(R)$. By (A2), $N(R) \leqslant N(P)$. But, as U is a Hall subgroup of $N(P)$, $R \in S_r(N(P))$. So $R \in$

$S_r(N(R))$. We conclude that $R \in S_r(G)$, in contradiction with 4.38(i). So $C(R) \nleqslant N(P)$.

Suppose that $d(R) \geqslant 2$. Then, by 4.37(v), there is an $x \in R^{\#}$ such that $d(C_P(x)) \geqslant 2$. But now, by 4.38(ii), $C(R) \leqslant C(x) \leqslant N(P)$, a contradiction. So $d(R) = 1$ and we have (i).

Suppose that $d(C_P(U)) \geqslant 2$. Then $d(C_P(R)) \geqslant 2$. By 4.38(ii), $C(R) \leqslant N(R) \leqslant N(P)$, a contradiction. So $C_P(U)$ is cyclic and we have (ii).

Let $1 \neq X \leqslant U$. In order to prove (iii), we may suppose that X is a q-subgroup of U, some $q \in \pi(U)$. Let $Q \in S_q(U)$ such that $Q \geqslant X$. By (i), $C(X) \nleqslant N(P)$. By 4.38(ii), $C_P(X)$ is cyclic. Let $P^* = C_P(X)$ and $C = C_U(P^*)$. As $X \leqslant C$, $C_P(C) = P^*$. By (i) and 4.37(i), U is cyclic. So $C \trianglelefteq U$. Thus U normalises P^*. But $1 \neq C_P(U) \leqslant P^*$. Then $[W_C, U] \leqslant P^*$, a cyclic group. By (A4), $[W_C, U] = 1$. By (A2), as $P \ntrianglelefteq G$, $W_C \neq 1$. So $W_C = \Omega_1(P^*) \leqslant C_{P^*}(U)$. By Theorem 5.2.4 of [10], $U = C$. So $C_P(U) \leqslant C_P(X) = P^* \leqslant C_P(U)$. We conclude that $C_P(U) = P^*$. We have (iii) and the lemma.

We now show that $C_P(U) \neq 1$.

LEMMA 4.40. *Assume Hypothesis* 4.35. *Assume further that* $C_P(U) = 1$. *Then*

(i) $m(W) \geqslant 3$.

(ii) *Each W-invariant $3'$-subgroup of G lies in* $C(W)$.

(iii) *Let* $T \in S_2(U)$. *Then* $C_T(P) = 1$.

Proof. As $C_P(U) = 1$ and $P \neq 1$, (i) is immediate from (A4).

Let B be a W-invariant $3'$-subgroup of G. By (C3) and Theorem 6.2.4 of [10], $B \leqslant N(P)$. So $[B, W] \leqslant B \cap P = 1$, i.e. $B \leqslant C(W)$. We have (ii).

Let $T \in S_2(U)$ such that $C_T(P) \neq 1$. Let $N = N(C_T(P))$. Then $P \leqslant N$. By 4.37(i), $U \leqslant N$. So, by 4.37(ii) and (iii), $N \leqslant N(P)$. It follows that G satisfies the hypotheses of 4.2. But now $P \trianglelefteq G$, a contradiction. So $C_T(P) = 1$ and we have (iii) and the lemma.

LEMMA 4.41. *Assume Hypothesis* 4.35. *Assume further that* $C_P(U) = 1$. *Let* $T \in S_2(U)$. *Then*

(i) $d(T) = 1$.

(ii) *If t is an involution of T then* $C_P(t) = 1$.

(iii) *G has no element of order* 6 *and P is abelian.*

Proof. We apply 4.20. Suppose $d(T) \geqslant 2$. By 4.38(i), $T \notin S_2(G)$. By 4.40(i), 4.20 (a) holds. By 4.38(ii), 4.20 (b) holds. By (C3) and 4.18, 4.20 (c) holds. By 4.20(iii), there is an element $x \in T^{\#}$ such that $P \leqslant C(x)$. So $C_T(P) \neq 1$, in contradiction with 4.40(iii). We have (i).

Let t be an involution of T. Suppose $C_P(t) \neq 1$. If $d(C_P(t)) \geqslant 2$ then, by 4.38(ii), $C(t) \leqslant N(P)$. But, by (i), $\langle t \rangle = \Omega_1(T)$. So $N(T) \leqslant C(t) \leqslant N(P)$.

So $T \in S_2(N(T))$ and so $T \in S_2(G)$, in contradiction with 4.38(i). We deduce that $C_P(t)$ is cyclic.

As $\langle t \rangle = \Omega_1(T)$, by 4.37(i), $t \in Z(U)$. Let $D = W_{\langle t \rangle}$. Now $[D, U] \leqslant C_P(t)$, a cyclic group. By (A4), $[D, U] = 1$. But, by (A2), as $P \not\leqslant G$, $D \neq 1$.

But $C_P(U) = 1$, a contradiction. So $C_P(t) = 1$ and we have (ii).

By (ii) and Theorem 10.1.4 of [10], P is abelian. Let $g \in G$ such that $o(g) = 6$. Let $h = g^3$ and $k = g^2$. Then $o(h) = 2$ and $o(k) = 3$ and $[h, k] = 1$. We may suppose $k \in P$. By (C3), $h \in N(P)$. By (C1), as U is a Hall subgroup, $T \in S_2(N(P))$. So we may suppose $h \in T$. By (ii), $k \in C_P(h) = 1$, a contradiction. So G has no elements of order 6. We have (iii) and the lemma.

LEMMA 4.42. *Assume Hypothesis* 4.35. *Then*

(i) *$E(G)$ is simple and* $3 \in \pi(E(G))$.

(ii) *If* $C_P(U) \neq 1$ *then* $P = (P \cap E(G)) \cdot C_P(U)$. *If* $C_P(U) = 1$ *then* $d(P \cap E(G)) \geqslant 3$.

(iii) *Suppose that* $L_2(3^a) \lesssim G$. *Then* $a \leqslant 2$.

(iv) $C_P(U) \neq 1$.

(v) *S_4 is involved in $E(G)$.*

Proof. Set $E = E(G)$. If $3 \notin \pi(E)$ then, by (C3), 4.36(i) and Theorem 6.2.4 of [10], $E \leqslant N(P)$, a solvable group. So $E = 1$. But now, by 4.37(iii), $F^*(G) = 1$. So $G = 1$, a contradiction. So $3 \in \pi(E)$.

Let E_1 be a component of E such that $3 \in \pi(E_1)$. As $E_1 \trianglelefteq E \trianglelefteq G$, $P \cap E_1 \in S_3(E_1)$. But, by (C3), as $N(P)$ is solvable, $C(E_1) \leqslant C(P \cap E_1) \leqslant N(P)$ and $C(E_1)$ is solvable. So $E = E_1$. By 4.37(iii), E is simple. We have (i).

Let $T = P \cap E$ and $T^* = Z(J(T))$. Then, by (i), $T \in S_3(E)$ and $T^* \neq 1$. By (C3), $C(T^*) \leqslant N(P)$. So $W \leqslant C_P(T^*) = O_3(C(T^*)) \trianglelefteq N(T^*)$. By 4.11, $N_E(T^*) \leqslant N(P) \cap E \leqslant N_E(T^*)$. So $N_E(T) = N_E(T^*) = N(P) \cap E$.

By (i) and Theorem 8.3.1 of [10], $N_E(T)$ is not 3-nilpotent. As $PC(P)$ is 3-nilpotent, $N_E(T) \not\leqslant PC(P)$. By (C2), $\pi \cap \pi(N_E(T)) \neq \emptyset$. As $N_E(T) = N(P) \cap E \trianglelefteq N(P)$, $U \cap N_E(T)$ is a Hall π-subgroup of $N_E(T)$. So $U \cap N_E(T) \neq 1$. Set $V = U \cap N_E(T)$.

As $V \leqslant N(P)$, by Theorem 5.3.5 of [10], $P = [P, V] \cdot C_P(V)$. Also $[P, V] \leqslant P \cap E$. So $P = (P \cap E) \cdot C_P(V)$. If $C_P(U) \neq 1$, by 4.39(iii), $P = (P \cap E) \cdot C_P(U)$. So suppose that $C_P(U) = 1$ and $d(P \cap E) \leqslant 2$.

Now $[W, V] \leqslant P \cap E$. So $m([W, V]) \leqslant 2$. By (A4), $[W, V] = 1$. Let $1 \neq X \leqslant V$. Then $W \leqslant C_P(X)$ and so $d(C_P(X)) \geqslant 3$. By 4.38(ii), $N(X) \leqslant N(P)$. Let $r \in \pi(V)$ and $V_r \in S_r(V)$. Then $N_E(V_r) \leqslant N(P) \cap E = N_E(T)$. As V is a Hall π-subgroup of $N_E(T)$, $V_r \in S_r(N_E(V_r))$ and so $V_r \in S_r(E)$. But, by (C4), if $X \leqslant V_r$, $\pi(\mathrm{Aut}_E(X)) \subseteq \pi(\mathrm{Aut}_G(X)) = \{r\}$. By Theorem 7.4.5 (a) of [10], E is r-nilpotent, in contradiction with (i). As $V \neq 1$, we conclude that $d(P \cap E) \geqslant 3$. We have (ii).

Suppose that $L_2(3^a) \lesssim E$ and $a \geqslant 3$. Let $H \leqslant E$ such that $H \simeq L_2(3^a)$. Let $Q \in S_3(H)$ and $X \leqslant N_H(Q)$ such that X is cyclic of order $3^a - 1$. Clearly we may suppose that $Q \leqslant P$. But, by 4.11 and (C3), $W \leqslant C_P(Q) = O_3(C(Q)) \trianglelefteq N(Q)$. So, by (A2), $N(Q) \leqslant N(P)$. By (C2), X is a π-group. So, as $N(P)$ is solvable, we can suppose that $X \leqslant U$. As $a \geqslant 3$, $|X|$ is not a power of 2. Let $q \in \pi(X) - \{2\}$ and $X_q \in S_q(X)$. Then $\mathrm{Aut}_H(X_q) \simeq Z_2$. But, by (C4), as $X_q \leqslant U$, $\mathrm{Aut}_G(X_q)$ is a q-group. As $|\mathrm{Aut}_H(X_q)|$ divides $|\mathrm{Aut}_G(X_q)|$, we have a contradiction. We have (iii).

Recall that $T = P \cap E \in S_3(E)$. Let $x \in T^{\#}$. Then, by (C3), $C_E(x) \leqslant N(P) \cap E = N_E(T)$. So $N_E(T)$ is a strongly 3-embedded subgroup of E. As $N(P)$ is solvable, $N_E(T)$ is solvable.

Now suppose that $C_P(U) = 1$. By 4.41(iii), G has no elements of order 6 and T is abelian. By (ii), as $C_P(U) = 1$, $m(T) \geqslant 3$. So, by the main theorem of [25], we conclude that $E \simeq L_2(3^a)$, $a \geqslant 3$. This contradicts (iii). So $C_P(U) \neq 1$ and we have (iv).

Suppose that S_4 is not involved in E. By (i) and Theorem C of Glauberman [7], $E \simeq L_2(3^{2n+1})$; $L_2(q)$, $q \equiv 3, 5 (8)$; $U_3(2^n)$, $n \geqslant 2$; $L_2(2^n)$, $n \geqslant 2$; Janko's simple group of order 175,560 (J_1), or E is of Ree type. By (iii) and the discussion on p. 481 of [10], $E \not\simeq L_2(3^{2n+1})$ and E is not of Ree type. Also, if $E \simeq L_2(q)$, $q \equiv 3, 5$ (8), q is not a power of 3. As $N_E(T)$ is strongly 3-embedded, using McBride's list [20], we see that T is cyclic.

It follows that $N_E(T)$ has even order. Recall that $V = U \cap N_E(T)$ is a Hall π-subgroup of $N_E(T)$. By (C1), V has even order. If $C_E(T)$ has even order then $V \cap C_E(T)$ has even order. By (iv) and 4.39(iii), $T \leqslant C_P(U)$. By (ii), $P = C_P(U)$. But now 4.36(i) and 4.39(ii) contradict each other. So $C_E(T)$ has odd order.

Now, by (C3), $C_E(\Omega_1(T)) \leqslant N(P) \cap E = N_E(T)$. By Theorem 5.2.4 of [10], we see that $C_E(\Omega_1(T))$ has odd order, so that E has no elements of order 6. It follows that $E \not\simeq J_1$. It also follows that $N_E(T)$ has a Sylow 2-subgroup of order exactly 2. So there is an involution $t \in V$.

We also have that $C_E(t)$ has order prime to 3. Also $[W, t] \leqslant P \cap E = T$, a cyclic group. By (A4), $[W, t] = 1$. So W normalises $C_E(t)$. In any case, $C_E(t)$ contains a Sylow 2-subgroup of E, S, such that $S \leqslant C_E(t)$. So $C_S(W) \leqslant S \cap N(P) = S \cap N_E(T)$. As $N_E(T)$ has a Sylow 2-subgroup of order 2, it follows that $C_S(W) = \langle t \rangle$.

If S is elementary of order 4 then, as $C_S(W) \neq 1$, $S = C_S(W) = \langle t \rangle$, a contradiction. So S is not elementary of order 4 and $E \not\simeq L_2(q)$, $q \equiv 3, 5$ (8). Thus $E \simeq U_3(2^n)$ or $L_2(2^n)$.

As W normalises S, W normalises $N_E(S)$. Let $R \in S_3(N_E(S) \cdot W)$, $R \geqslant W$. Let $R^* = R \cap N_E(S) \in S_3(N_E(S))$. By 4.11 and 4.6(i), $R^* \leqslant P \cap E = T$. As E has no elements of order 6, $W \not\leqslant E$. Also $R = R^*W \leqslant P$.

Suppose that $R^* \neq 1$. It follows that R is not cyclic. So, by (C3) and Theorem 6.2.4 of [10], $S \leqslant N(P) \cap E = N_E(T)$. So, as $\langle t \rangle \in S_2(N_E(T))$, $S = \langle t \rangle \simeq Z_2$, a contradiction. So $R^* = 1$ and $N_E(S)$ has order prime to 3.

It follows that W normalises a Hall subgroup X of $N_E(S)$ of order $2^n - 1$. Also X acts regularly on $\Omega_1(S)^\#$. It follows that XW acts faithfully on $\Omega_1(S)$, an elementary abelian 2-group of rank n. Also $C_{\Omega_1(S)}(W) = \langle t \rangle \cong Z_2$. So, by 2.17, $n = 3$.

If $E \cong U_3(2^n)$ then $E \cong U_3(8)$. But now 3 divides $|N_E(S)|$, a contradiction. We conclude that $E \cong L_2(8)$. By 4.37(iii), $F(G) = 1$. So $F^*(G) = E$ and $G \lesssim \mathrm{Aut}(E) \cong P\Gamma L(2, 8)$. As $P \geqslant TW$ and $W \nleqslant E$, P is isomorphic to a Sylow 3-subgroup of $\Gamma L(2, 8)$, a non-abelian group of order 3^3. Also $P = TW$. By 4.11, P is abelian. This is a contradiction. So E involves S_4 and we have (v) and the lemma.

LEMMA 4.43. *Assume Hypothesis 4.35 and let $C = C_P(U)$. Then the following holds.*

(i) *Let $1 \neq P_0 \leqslant P$. Then $N(P_0) \leqslant N(P)$.*

(ii) $2 \in \pi(U)$.

(iii) *Let t be an involution of U. Then $C(t)$ is 3-nilpotent and $C \in S_3(C(t))$.*

(iv) *Let $a \in P$ such that $a^{N(P)} \cap C = \emptyset$. Then $C(a) \leqslant PC(P)$.*

(v) *There is a Sylow 2-subgroup S of G such that $N_S(P) \in S_2(U)$ and $C \leqslant N(S)$.*

Proof. Let $1 \neq P_0 \leqslant P$. By (C3), $W \leqslant C_P(P_0) = O_3(C(P_0)) \trianglelefteq N(P_0)$. By (A2), $N(P_0) \leqslant N(P)$. We have (i).

Suppose that $2 \notin \pi(U)$. Then, by (B1), $N(P)$ has odd order. By 4.42(v), S_4 is involved in G. Let $H, K \leqslant G$ such that $K \trianglelefteq H$ and $H/K \cong S_4$. Let $L = O_{3',3}(H \bmod K)$. Let $P_0 \in S_3(L)$. Then $P_0 \neq 1$. By the Frattini argument, $H = LN(P_0)$. We may suppose $P_0 \leqslant P$.

By (i), $N(P_0) \leqslant N(P)$ and so $N(P_0)$ has odd order. So $|H : L|$ is odd. But $|H : L| = 2$, a contradiction. So $2 \in \pi(U)$ and we have (ii).

Let t be an involution of U. By 4.42(iv), $C \neq 1$. By (i), $C \in S_3(N_{C(t)}(C))$ and so $C \in S_3(C(t))$. As C is cyclic, $|N_{C(t)}(C) : C_{C(t)}(C)| \leqslant 2$. But now, by (C1), as $U \leqslant C(C_P(U)) \leqslant N(C_P(U)) \leqslant N(P)$, $|N(C_P(U)) : C(C_P(U))|$ is odd. So $C \leqslant Z(N_{C(t)}(C))$. By Burnside's transfer theorem, $C(t)$ is 3-nilpotent. We have (iii).

Let $a \in P$ such that $a^{N(P)} \cap C = \emptyset$. Suppose that $q \in \pi$ and $Q \in S_q(C(a))$ such that $Q \neq 1$. We may suppose $Q \leqslant U$. But now, by 4.39(iii), $a \in C_P(Q) \leqslant C$, a contradiction. So $C(a)$ is a π'-group. By (C2) and (C3), $C(a) \leqslant PC(P)$. We have (iv).

Let $T \in S_2(U)$ and S be a maximal 2-subgroup subject to $S \geqslant T$ and $C \leqslant N(S)$. Let $V \in S_3(N(S))$ such that $V \geqslant C$. By (i), $N_V(V \cap N(P)) \leqslant V \cap N(P)$. So $V \leqslant N(P)$. By (ii), $T \neq 1$. Also $[V, T] \leqslant S \cap P = 1$. So $V \leqslant C_P(T)$. By 4.39(iii), $V \leqslant C_P(U) = C$. So $V = C \in S_3(N(S))$.

By (i), $C \leqslant Z(N_{N(S)}(C))$. So, by Burnside's Transfer theorem, $N(S)$ is

3-nilpotent. Let $S^* \in S_2(O_{3'}(N(S)))$ such that $C \leqslant N(S^*)$. Then $S^* \geqslant S \geqslant T$. By maximality of S, $S^* = S \in S_2(N(S))$. So $S \in S_2(G)$.

Certainly $C \leqslant N(S)$. Also $T = N_S(P) \in S_2(U)$. We have (v) and the lemma.

Set $C = C_P(U)$. For $a \in P$ such that $a^{N(P)} \cap C = \emptyset$, let $\mathscr{S}_a$ be the collection of non-identity 2-subgroups, R, of G such that $a \in N(R)$. We show that, for some a, $\mathscr{S}_a \neq \emptyset$. We then consider a maximal element A of $\mathscr{S}_a$. We show that, if $\sigma = \{2, 3\}$, $M = O^{\sigma'}(N(A))$ and $K = O_\sigma(N(A))$ and, for $H \leqslant N(A)$, $\bar{H} = HK/K$, then $\bar{M}/\bar{A} \cong SL(2, 2^n)$, $n \geqslant 2$. A standard "pushing up" argument gives us the required contradiction.

LEMMA 4.44. *Assume Hypothesis* 4.35. *Then, for some $a \in P$ such that $o(a) = 3$ and $a^{N(P)} \cap C = \emptyset$, there is an $R \in \mathscr{S}_a$ such that $N(R)$ involves S_4 and $R \nleqslant N(P)$.*

Proof. By 4.42(v), there are subgroups H, K of G such that $K \trianglelefteq H$ and $H/K \cong S_4$. Let $L = O_2(H \bmod. K)$ and $R \in S_2(L)$. Then $R \neq 1$. Also, by the Frattini argument,

$$H = LN_H(R) = KRN_H(R) = KN_H(R).$$

So $N_H(R)/N_K(R) \cong S_4$ and we may suppose $R \trianglelefteq H$. Clearly we may also suppose $P \cap H \in S_3(H)$.

Suppose $R \leqslant N(P)$. By (C1), we may suppose $R \leqslant U$. By 4.39(i) and 4.43(iii), $H = N(R) \leqslant C(\Omega_1(R))$, a 3-nilpotent group, a contradiction, as H involves S_4. So we have that

$$R \nleqslant N(P) \text{ and } N(R) \text{ involves } S_4.$$

Set $Q = P \cap H$ and suppose that $d(Q) \geqslant 2$. By (C3) and Theorem 6.2.4 of [10], $R \leqslant N(P)$, a contradiction. So Q is cyclic. Clearly $Q \neq 1$. Let $a \in \Omega_1(Q)^{\#}$ and suppose that $a^{N(P)} \cap C \neq \emptyset$. Then we can suppose $a \in C$.

By (C1) and 4.43(i), a is not conjugate to a^{-1}. So $Q \leqslant Z(N_H(Q))$. By Burnside's Transfer theorem, H is 3-nilpotent. But H involves S_4. This is a contradiction. We deduce that $a^{N(P)} \cap C = \emptyset$. So $R \in \mathscr{S}_a$ and the lemma follows.

Fix $a \in P$ such that $o(a) = 3$ and $a^{N(P)} \cap C = \emptyset$ and fix $R \in \mathscr{S}_a$ such that $R \nleqslant N(P)$ and $N(R)$ involves S_4. Let A be a maximal member of $\mathscr{S}_a$ with respect to inclusion such that $A \geqslant R$.

LEMMA 4.45. *Assume Hypothesis* 4.35. *Let $Q \in S_3(N(A))$ such that $a \in Q$. Let $T \in S_2(N(A))$. Then*

(i) *$Q \leqslant P$.*

(ii) *Q is cyclic.*

(iii) *A is not a characteristic subgroup of T.*

(iv) *$A \in S_2(O_{3'}(N(A)))$.*

(v) *$N_{N(A)}(Q) \gneqq C_{N(A)}(Q)$.*

Proof. Now $Q \cap P \neq 1$. By 4.43(i), $N_Q(Q \cap P) \leqslant Q \cap N(P) = Q \cap P$. So $Q \leqslant P$. We have (i).

Suppose that $d(Q) \geqslant 2$. By (C3) and Theorem 6.2.4 of [10], $R \leqslant A \leqslant N(P)$, a contradiction as $R \nleqslant N(P)$. So Q is cyclic and we have (ii).

Suppose that A is a characteristic subgroup of T. Then $N(T) \leqslant N(A)$. So $T \in S_2(N(T))$ and $T \in S_2(G)$. By 4.43(v), there is a $g \in G$ such that $C^g \leqslant N(T) \leqslant N(A)$. So there is an $n \in N(A)$ such that $C^{gn} \leqslant Q$. By 4.42(iv), $U \in \mathscr{X}$. So $Wt^n \leqslant C^{gn} \leqslant Q \leqslant P$. By (A2), $gn \in N(P)$. Also $\langle a \rangle = \Omega_1(Q)$. As $C \neq 1$, $a \in C^{gn}$ and $a^{N(P)} \cap C \neq \emptyset$, a contradiction. So A is not a characteristic subgroup of T. We have (iii).

Let $A^* \in S_2(O_{3'}(N(A)))$ such that $Q \leqslant N(A^*)$. Then $A^* \geqslant A \geqslant R$. By maximality of A, $A^* = A$. We have (iv).

Suppose that $N_{N(A)}(Q) = C_{N(A)}(Q)$. Then, by Burnside's Transfer theorem, $N(A) = O_{3'}(N(A))Q$. By (iv), we have that $A \in S_2(N(A))$. This contradicts (iii). So $N_{N(A)}(Q) \gneqq C_{N(A)}(Q)$ and we have (v) and the lemma.

LEMMA 4.46. *Assume Hypothesis* 4.35. *Let* $T \in S_2(U)$. *Then* $C_T(P) = 1$.

Proof. Assume that $C_T(P) \neq 1$. By 4.37(i), $C_T(P) \trianglelefteq U$. So $PU \leqslant N(C_TP))$. By 4.37(ii) and (iii), $N(C_T(P)) \leqslant N(P)$. So $C(T) \leqslant N(C_T(P)) \leqslant N(P)$. By 4.42(iv), $C \neq 1$. But now, by 4.39(i), we have a contradiction. So $C_T(P) = 1$ and the lemma is proved.

For $H \leqslant G$, let $S(H)$ denote the largest solvable normal subgroup of H. Let $\sigma = \{2, 3\}$ and $M = O^{\sigma'}(N(A))$. Let $K = O_{\sigma'}(N(A))$ and $Q \in S_3(N(A))$ such that $a \in Q$.

LEMMA 4.47. *Assume Hypothesis* 4.35. *Then*

(i) $C(a)$ *and* $C(Q)$ *have odd order.*

(ii) $N(A)$ *is not solvable.*

Proof. By 4.43(iv), $C(a) \leqslant PC(P)$. By (C1) and 4.46, $C(P)$ has odd order. So $C(a)$, and hence $C(Q)$, have odd order. We have (i). Now suppose that $N(A)$ is solvable.

Let $N = N(A)$. By 4.45(v), $N_N(Q) > C_N(Q)$. By 4.45(ii) and (i), $N_N(Q) = C_N(Q) \cdot \langle t \rangle$ where $o(t) = 2$. By Theorem 6.3.3 of [10], $Q \leqslant O_{3',3}(N)$. As $C_N(Q)$ is 3-nilpotent, by 4.5 and the Frattini argument,

$$N = O_{3',3}(N) \cdot N_N(Q) = O_{3'}(N) \cdot N_N(Q) = O_{3'}(N) \cdot Q \cdot \langle t \rangle.$$

By 4.45(iv), $A \in S_2(O_{3'}(N))$. We deduce that $A \cdot \langle t \rangle \in S_2(N)$. By 4.45(iii), A is not a characteristic subgroup of $A \cdot \langle t \rangle$.

Consider $A \cdot \langle a, t \rangle$. As $C_A(a) = 1$ and $\langle a, t \rangle \cong D_6$, by Theorem 8.1 of [16], $A \cong V_4$. So $A \cdot \langle t \rangle \cong D_8$. Let $S \in S_2(G)$ such that $S \geqslant A \cdot \langle t \rangle$. Then

$C_S(A) = A$. By a well-known result (cf. p. 35 of [16]), S is dihedral or quasi-dihedral.

By 4.43(v), there is a $g \in G$ such that C^g normalises S. So $C^g \leqslant C(S)$. By (C3), if $h = g^{-1}$, $R^h \leqslant N(P)$. So there is an $n \in N(P)$ such that $R^{hn} \leqslant U$. But, by 4.39(i) and 4.43(iii), $N(R^{hn}) \leqslant C(\Omega_1(R^{hn}))$, 3-nilpotent group. But $N(R)$ involves S_4, a contradiction. So N is not solvable and we have (ii) and the lemma.

LEMMA 4.48. *Assume Hypothesis* 4.35. *Then*

(i) $S(M) = O_{3'}(M)$.

(ii) *$C_M(Q)$ is a cyclic Hall subgroup of M. Also $C(x) \leqslant N(P)$ for each $x \in C_M(Q)^{\#}$.*

(iii) *$N_M(Q)$ is dihedral of twice odd order.*

(iv) $N_M(Q) \cap S(M) = 1$.

(v) *$S(M)$ is nilpotent. In particular $S(M) = K \times A$.*

Proof. By 4.47(ii) and the Feit-Thompson theorem [5], M is not solvable. Let $Q_0 = Q \cap S(M)$. Then $Q_0 \in S_3(S(M))$. Suppose that $Q_0 \neq 1$. By the Frattini argument, 4.45(i) and 4.43(i),

$$M = S(M)\, N_M(Q_0) = S(M) \cdot N_M(P).$$

As $N(P)$ is solvable, M is solvable, a contradiction. So $Q_0 = 1$ and $S(M) \leqslant O_{3'}(M)$.

Now $O_{3'}(M) \leqslant O_{3'}(N(A))$. By 4.45(iv), $A \in S_2(O_{3'}(M))$. So, by the Feit-Thompson theorem, $O_{3'}(M)/A$ is solvable. So $O_{3'}(M) \leqslant S(M)$ and we have (i).

Let $x \in C_M(Q)^{\#}$. We show that $C(x) \leqslant N(P)$. By (C3) and 4.45(i), $x \in N(P)$. We can suppose that x is a π-element or x is a π'-element. Also, by (C3), we can suppose that $3 \nmid o(x)$.

If x is a π'-element then, by (C2), $x \in PC(P)$. By 4.37(iii), $C(x) \lneqq G$. We can suppose $C_U(x)$ is a Hall π-subgroup of $N_{C(x)}(P)$. As x is a 3'-element and $x \in PC(P)$, $x \in C(P)$. So, by 4.37(ii), as $P \leqslant C(x)$, $C(x) \leqslant N(P)$.

If x is a π-element then there is an $n \in N(P)$ such that $x^n \in U$. But, by 4.42(iv) and 4.39(iii), $Q^n \leqslant C_P(x^n) = C$. But now $a^{N(P)} \cap C \neq \emptyset$, in contradiction with choice of a. We deduce that, in all cases, of $x \in C_M(Q)^{\#}$ then $C(x) \leqslant N(P)$.

Let $r \in \pi(C_M(Q))$ and $T \in S_r(C_M(Q))$. Then

$$Q \leqslant C_M(T) \leqslant N(P) \cap M \leqslant N_M(Q).$$

By the Frattini argument, as $Q \in S_3(C_M(T))$,

$$N_M(T) \leqslant N_M(Q). \tag{19}$$

By 4.45(ii) and 4.45(v),

$$|N_M(Q) : C_M(Q)| = 2. \tag{20}$$

By 4.47(i), r is odd. So, by (19) and (20), $T \in S_r(N_M(T))$ and so $T \in S_r(M)$. We deduce that

$$C_M(Q) \text{ is a Hall subgroup of } M. \tag{21}$$

Suppose that $d(T) \geqslant 2$. Then, by Theorem 6.2.4 of [10], $A \leqslant N(P)$. But now $R \leqslant N(P)$, in contradiction with the choice of R. So, by 4.47(i),

$$C_M(Q) \text{ has cyclic Sylow subgroups.} \tag{22}$$

Set $N = N_M(Q)$. By (i) and (20), $N_M(Q)$ has cyclic Sylow subgroups. By Theorem 7.6.2 of [10], $N/F(N)$ is cyclic. Let $r \in \pi(N) - \sigma$ and $T \in S_r(N)$. By (20), $T \in S_r(C_M(Q))$. By (19) and and Theorem 7.5.2 of [10], as $M = O^{\sigma'}(M)$, $N = O^r(N)$. We deduce that $T \leqslant F(N)$.

By (20), there is an involution $t \in N_M(Q)$. But, by 4.45(ii) and 4.47(i), t inverts Q. So $Q = [Q, t] \leqslant F(N)$. By 4.47(i), $C_M(Q)$ has odd order. We conclude that $C_M(Q)$ is cyclic. By (21), we have (ii).

As $|N{:}F(N)| = 2$, $N = O^r(N)$ for each $r \in \pi(N) - \sigma$ and $Q = [Q, t] \leqslant N'$, we deduce that $N' = F(N)$. As $F(N)$ is cyclic, t acts fixed-point-freely on $F(N)$ and $N = F(N) \cdot \langle t \rangle$ is dihedral. By 4.47(i), we have (iii).

Let $r \in \pi(N_M(Q) \cap S(M))$ and $T^* \in S_r(N_M(Q) \cap S(M))$. Then, by (i), $[T^*, Q] \leqslant Q \cap O_{3'}(M) = 1$. So $T^* \leqslant C_M(Q)$. By (ii),

$$Q \leqslant C_M(T^*) \leqslant N_M(Q).$$

So, by the Frattini argument, $N_M(T^*) \leqslant N_M(Q)$. By (ii), $T^* \in S_r(S(M))$. But now, by the Frattini argument,

$$M = S(M) \cdot N_M(T^*) = S(M) \cdot N_M(Q).$$

But now, by (iii), M is solvable, in contradiction with (i). So $N_M(Q) \cap S(M) = 1$ and we have (iv).

Now, by (C3) and (iv),

$$C_{S(M)}(a) \leqslant S(M) \cap N(P) \leqslant S(M) \cap N_M(P \cap M) = S(M) \cap N_M(Q) = 1.$$

By Theorem 10.2.1 of [10], as $o(a) = 3$, $S(M)$ is nilpotent. By (i), $S(M) = O_{3'}(M) \leqslant O_{3'}(N(A))$. By 4.45(iv), $A \in S_2(O_{3'}(M))$. So $A \in S_2(S(M))$. We conclude that $S(M) = K \times A$. We have (v) and the lemma.

LEMMA 4.49. *Assume Hypothesis* 4.35. *Then* $M/S(M) \simeq SL(2, 2^n)$, $n \geqslant 2$.

Proof. For $H \leqslant M$, set $\bar{H} = HS(M)/S(M)$. Then $S(\bar{M}) = F(\bar{M}) = 1$. By 4.48(i), $\bar{M} \neq 1$ and so $E(\bar{M}) \neq 1$. Let $N = E(M \bmod. S(M))$. By 4.48(i), $3 \in \pi(\bar{N})$. Let $\bar{N}^*$ be a component of $\bar{N}$ such that $3 \in \pi(\bar{N}^*)$. Let N^* be the pre-image in M of $\bar{N}^*$. Then $N^* \trianglelefteq N \trianglelefteq M$. It follows that $Q \cap N^* \neq 1$. By 4.43(i) and 4.45(i), as $N(P)$ is solvable, $N_N(Q \cap N^*)$ is solvable. By the Frattini argument, $N = N^* \cdot N_N(Q \cap N^*)$. So N/N^* is solvable. As $N = E(M)$, $\bar{N} = \bar{N}^*$, and $N = N^*$. As $S(\bar{M}) = 1$, $\bar{N}$ is simple. So $Q \cap N = Q \cap N^*$ and, by 4.43(i) and 4.45(i),

$$N_M(Q \cap N) \leqslant M \cap N(P) \leqslant N_M(Q). \tag{23}$$

Clearly N is not 3-nilpotent. By Burnside's Transfer theorem, $N_N(Q \cap N) \gneqq C_N(Q \cap N)$. By 4.45(ii) and (23), there is an involution $t \in N(Q) \cap N$. By 4.45(ii) and 4.47(i), $C_Q(t) = 1$. So, as $N \trianglelefteq M$,

$$Q = [Q, t] \leqslant N.$$

By the Frattini argument, $M = N \cdot N(Q)$. As $Q \cdot \langle t \rangle \leqslant N_N(Q)$

$$| M : N | = | N_M(Q) : N_N(Q)| \,/\, | N_M(Q) : Q \cdot \langle t \rangle|.$$

By 4.48(iii), $| M : N |$ is prime to 2 and 3. As $M = O^{\sigma'}(M)$, we deduce that

$$M = N. \tag{24}$$

So $\bar{M}$ is a simple group. Now $\bar{Q}$ is a cyclic Sylow 3-subgroup of $\bar{M}$. Let $X \leqslant Q$ such that $\bar{X} \neq 1$ and L be the pre-image in M of $N_{\bar{M}}(\bar{X})$. By 4.48(i), $X \in S_3(S(M) \cdot X)$. By the Frattini argument, as $S(M) \cdot X \trianglelefteq L$,

$$L = S(M) \cdot X \cdot N_L(X) = S(M) \cdot N_L(X).$$

But, by 4.45(i) and 4.43(i),

$$N_L(X) \leqslant N(P) \cap L \leqslant N_M(Q) \cap L = N_L(Q).$$

We deduce that

$$N_{\bar{M}}(\bar{X}) = \overline{N_L(X)} \leqslant \overline{N_M(Q)} \leqslant N_{\bar{M}}(\bar{Q}) \tag{25}$$

So $N_{\bar{M}}(\bar{Q}) = \overline{N_M(Q)}$. By 4.48(iii) and (iv), $N_{\bar{M}}(\bar{Q})$ is dihedral of twice odd order. By (25), $C_{\bar{M}}(\bar{a}) \leqslant N_{\bar{M}}(\bar{Q})$. So $C_{\bar{M}}(\bar{a})$ has odd order. Also there is an involution $\bar{t}$ such that $\bar{t} \in N_{\bar{M}}(\bar{Q})$ and $\bar{t}$ inverts $C_{\bar{M}}(\bar{a}) = C_{\bar{M}}(\bar{Q})$. As $\bar{M}$ is simple, by Theorem 2.2 of [17], $\bar{M} \simeq SL(2, 2^n)$, $n \geqslant 2$ or $\bar{M} \simeq PSL(2, q)$, $q \equiv \pm 5(12)$.

Suppose that $\bar{M} \simeq PSL(2, q)$, $q \equiv \pm 5(12)$. Then q is odd. Let $\bar{H} \leqslant \bar{M}$ such that $\bar{a} \in \bar{H}$ and $\bar{H} \simeq A_4$. Let H be the pre-image in $\bar{M}$ of $\bar{H}$. Let $H^* =$

$O_2(H \bmod . S(M))$ and A^* be an a-invariant Sylow 2-subgroup of H^*. Then $A^* \geqslant A \geqslant R$. But now, by maximality of A, $A^* = A$. So $H^* = S(M)A^* = S(M)$, a contradiction. So $\bar{M} \simeq SL(2, 2^n)$, $n \geqslant 2$. The lemma is proved.

For $H \leqslant M$, let $\bar{H} = HK/K$. For a p-group X, let $J_e(X)$ be the group generated by all elementary abelian subgroups A of X such that $m(A) = d(X)$. We now prove

LEMMA 4.50. *Assume Hypothesis* 4.35. *Suppose that* $M/S(M) \simeq SL(2, 2^n)$, $n \geqslant 2$. *Then A is elementary abelian of rank $2n$. Let $T \in S_2(N(A))$. Then there is a normal elementary abelian subgroup B of T such that $m(B) = 2n$ and the following holds.*

(i) $T = AB$.

(ii) *$A \cap B$ has rank n.*

(iii) $A \cap B = Z(T)$.

(iv) *Every involution of T lies in $A \cup B$.*

Proof. We claim that $C_{\bar{A}}(\bar{a}) = 1$. Let $A^* \leqslant A$ such that $\bar{A}^* = C_{\bar{A}}(\bar{a})$. Then A^* normalises $K \cdot \langle a \rangle$. We deduce that $[A^*, a] \leqslant A \cap K \cdot \langle a \rangle = 1$. So, by 4.47(i), $A^* = 1$ and so $C_{\bar{A}}(\bar{a}) = 1$.

By 4.49 and 4.48(iv), $\bar{M}/\bar{A} \simeq SL(2, 2^n)$. By Theorem 8.2 of [16], $\bar{A}$ is a direct sum of natural $SL(2, 2^n)$-modules. In particular $\bar{A}$ is elementary abelian.

As K has odd order, a Sylow 2-subgroup of M is isomorphic to a Sylow 2-subgroup of $\bar{M}$. It follows from 4.45(iii) that $\bar{A}$ is not a characteristic subgroup of $\bar{S}$ for any $\bar{S} \in S_2(\bar{M})$. Note that $T \leqslant M$ and $A \simeq \bar{A}$.

But, if $m(\bar{A}) \geqslant 2n$, by Theorem 8.2 of [16], $\bar{A}$ is a characteristic subgroup of $\bar{T}$, a contradiction. So, as $A \simeq \bar{A}$, $m(A) = m(\bar{A}) = 2n$. As $\bar{A}$ is not a characteristic subgroup of $\bar{T}$ there is a $\phi \in \operatorname{Aut}(\bar{T})$ such that $\bar{A}^\phi \neq \bar{A}$. Set $\bar{B} = \bar{A}^\phi$. Then $\bar{B}\bar{A}/\bar{A} \neq 1$. Also, as $\bar{M}/\bar{A}$ acts naturally on $\bar{A}$ and $\bar{A} \cap \bar{B}$ is centralised by $\bar{B}\bar{A}/\bar{A}$, $\bar{A} \cap \bar{B} \leqslant Z(\bar{T}) \simeq (\mathbb{Z}_2)^n$. Let $B \leqslant T$ such that $\bar{B} = BK/K$. Then $B \simeq \bar{B}$ and so B is elementary abelian of rank $2n$. Also

$$A \cap B \simeq \overline{A \cap B} \leqslant \bar{A} \cap \bar{B} = Z(\bar{T}) \simeq (\mathbb{Z}_2)^n.$$

So $|A \cap B| \leqslant 2^n$. As $T \simeq \bar{T}$, $|T| = 2^{3n}$. By considering $|AB|$ we see that $T = AB$ and $A \cap B \simeq (\mathbb{Z}_2)^n$. Now $\overline{Z(T)} \leqslant Z(\bar{T}) \simeq (\mathbb{Z}_2)^n$. As $A \cap B \leqslant Z(T)$, $A \cap B = Z(T) \simeq (\mathbb{Z}_2)^n$. We have proved (i), (ii), and (iii).

Let t be an involution of T. By (i), $t = cb$ for $c \in A$, $b \in B$. As $o(c)$ and $o(b) \leqslant 2$, $[c, b] = 1$. In order to prove (iv), we may suppose $b \notin A$. So $\bar{b} \notin \bar{A}$. But $\bar{A}$ is a natural $\bar{M}/\bar{A}$ module. So $\bar{A} \cap \bar{B} = C_{\bar{A}}(\overline{bA}/\bar{A})$. We deduce that $\bar{c} \in \bar{A} \cap \bar{B}$. So

$$c \in KB \cap T = B(K \cap T) = B.$$

So $t = cb \in B$. We have (iv) and the lemma.

LEMMA 4.51. *Assume Hypothesis* 4.25. *Then there is a* $g \in G$ *such that* $C^g \leqslant N(A)$.

Proof. Let $T \in S_2(N(A))$ and n, B be as in 4.50. Let $S^* \in S_2(G)$ such that $S^* \geqslant T$ and set $S = N_{S^*}(T)$. Let X be an elementary abelian subgroup of T. We claim that either $X \leqslant A$ or $X \leqslant B$.

Suppose that $X \not\leqslant A$. By 4.50(iv), there is a $b \in (B \cap X) - A$. Let $x \in X \cap A$ Then xb is an involution. By 4.50(iv), $xb \in A$ or $xb \in B$. If $xb \in A$ then, as $x \in A$, $b \in A$, a contradiction. So $xb \in B$. But now $x \in B$. So $X \cap A \leqslant B$. By 4.50(iv), $X \leqslant B$. The claim is justified.

So S permutes $\{A, B\}$. So $|S : T| = |S : N_S(A)| \leqslant 2$. If $S = T$ then $S^* = T \in S_2(G)$. By 4.43(v), there is a $g \in G$ such that $C^g \leqslant N(T)$. So C^g permutes $\{A, B\}$. As C^g is a 3-group, $C^g \leqslant N(A)$.

We may therefore suppose $|S : T| = 2$. Let E be an elementary abelian subgroup of such that $m(E) = d(S)$. Suppose that $E \not\leqslant T$. Let $e \in E - T$.

Now, as $|S : T| = 2$, $|E : E \cap T| = 2$. As $e \notin N(A)$, $A^e = B$. Also $E \cap T \leqslant A$ or $E \cap T \leqslant B$. But $E \cap T \leqslant C_T(e)$ and so as $A^e = B$, $E \cap T \leqslant A \cap B$. By 4.50(ii), $m(E \cap T) \leqslant n$. So

$$2n = m(A) \leqslant d(S) = m(E) = m(E \cap T) + 1 = n + 1.$$

So $n \leqslant 1$. But $n \geqslant 2$, a contradiction. We deduce that $E \leqslant T$. We conclude that, by 4.50(i), $J_e(S) = J_e(T) = AB = T$. So $T \trianglelefteq N_S{}^*(S)$ and $N_S{}^*(S) = S$. Thus $S^* = S \in S_2(G)$.

By 4.43(v), there is a $g \in G$ such that

$$C^g \leqslant N(S) \leqslant N(J_e(S)) = N(J_e(T)) = N(T).$$

So C^g permutes $\{A, B\}$. As C is a 3-group, $C^g \leqslant N(A)$. The lemma is proved.

Proof of 4.3. By 4.51, there is a $g \in G$ such that $C^g \leqslant N(A)$. So there is an $n \in N(A)$ such that $C^{gn} \leqslant Q$. By 4.42(iv), $C \neq 1$ and $a \in C^{gn}$. Also $U \in \mathscr{X}$. By 4.45(i), $W_{U^{gn}} \leqslant Q \leqslant P$. So, by (A2), $gn \in N(P)$. But now $a^{N(P)} \cap C \neq \emptyset$, in contradiction with the choice of a. So no minimal counterexample to 4.3 can exist. We have proved 4.3.

We now prove 4.4. We therefore introduce:

HYPOTHESIS 4.52. G is a minimal counterexample to 4.4.

We prove 4.4 in essentially the same way that we proved 4.2 and 4.3. Let $W = W_1$ (note $1 \in \mathscr{X}$). Observe that $U \in S_2(N(P))$.

LEMMA 4.53. *Assume Hypothesis* 4.52. *Then* $Z(G) = 1$.

Proof. Let $q \in \pi(G)$ and $H = O_q(Z(G))$. Choose W_X in accordance with 4.15 and assume the notation of 4.9. Clearly $H \leqslant N(P)$ and $P \not\leqslant H$.

By 4.9(i), $\bar{G}$, $\bar{P}$, $\bar{U}$, $W_{\bar{X}}$, π satisfy Hypothesis 4.1. Also $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$. By choice of W_X, $W_{\bar{1}} = \bar{W}$. So, by (D3) and 4.9(ii), (D3) holds for $\bar{G}$.

Clearly (D1) holds for $\bar{G}$. Also, by 4.9(i) and (D2),

$$O^2(N_{\bar{G}}(\bar{P})) = \overline{O^2(N(P))} \leqslant \overline{PC(P)} \leqslant \bar{P}C_{\bar{G}}(\bar{P}).$$

As $\bar{U} \in S_2(\overline{N(P)}) = S_2(N_{\bar{G}}(\bar{P}))$, (D2) holds for $\bar{G}$.

Let $\bar{V}$ be a 2-subgroup of $\bar{U}$, V be the pre-image in G of $\bar{V}$ and $V_2 \in S_2(V)$. By 4.21(i), $N_{\bar{G}}(\bar{V}) = \overline{N(V_2)}$. So $O^2(N_{\bar{G}}(\bar{V})) = \overline{O^2(N(V_2))}$. Now, as $H \leqslant N(P)$, $V \leqslant N(P)$. As $V \leqslant HU$ and $U \in S_2(N(P))$, it follows that $V_2 \leqslant U$. So, by (D4),

$$O^2(N_{\bar{G}}(\bar{V})) = \overline{O^2(N(V_2))} \leqslant \overline{C(V_2)} \leqslant C_{\bar{G}}(\bar{V}_2) = C_{\bar{G}}(\bar{V}).$$

So (D4) holds for $\bar{G}$.

We conclude that $\bar{G}$ satisfies the hypotheses of 4.4. As $H \leqslant N(P)$, $\bar{P} \ntrianglelefteq \bar{G}$. By minimality of G, $H = 1$. We deduce that $Z(G) = 1$. The lemma is proved.

LEMMA 4.54. *Assume Hypothesis 4.52. Let $H \lneqq G$ such that $W_X \leqslant H$ for each $X \in \mathscr{X}$. Assume further that $U \cap H$ is a Sylow 2-subgroup of $N_H(P)$. Then $H \leqslant N(P)$.*

Proof. Set $Q = P \cap H$. By 4.12, H, Q, $U \cap H$, W_X, π satisfy Hypothesis 4.1. Also $N_H(P) = N_H(Q)$. By (D2), as $PC(P)$ is 3-nilpotent,

$$O^2(N_H(Q)) \leqslant O^2(N(P)) \cap H \leqslant PC(P) \cap H \leqslant QC_H(Q).$$

So (D2) holds for H. We conclude that H satisfies the hypotheses of 4.4. As $H \lneqq G$, by minimality of G, $Q \trianglelefteq H$ and $H \leqslant N(P)$. The lemma is proved.

LEMMA 4.55. *Assume Hypothesis 4.52. Then*

(i) *$C(a) \leqslant N(P)$ for each $a \in Z(P)^{\#}$.*

(ii) *$F(G) = 1$.*

(iii) *There is an $a \in P^{\#}$ such that $C(a) \nleqslant N(P)$.*

Proof. Let $a \in Z(P)^{\#}$. In order to show $C(a) \leqslant N(P)$, we may suppose that $C_U(a) \in S_2(C_{N(P)}(a))$. Also $P \leqslant C(a)$. So, by 4.53 and 4.54, $C(a) \leqslant N(P)$. We have (i).

Suppose that $O_3(G) \neq 1$. Then, as $O_3(G) \leqslant P$, $O_3(G) \cap Z(P) \neq 1$. By (i), $C(O_3(G)) \leqslant C(O_3(G) \cap Z(P)) \leqslant N(P)$. By 4.10, $P \trianglelefteq G$, a contradiction. So $O_3(G) = 1$.

Let $q \in \pi(F(G))$. By (D3) and 4.11, $O_q(G) \leqslant N(P)$. So, as $q \neq 3$, $P \leqslant C(O_q(G))$. As $N(P) \cap C(O_q(G)) \trianglelefteq N(P)$, $U \cap C(O_q(G))$ is a Sylow 2-subgroup of $N(P) \cap C(O_q(G))$. So, by 4.53 and 4.54, $C(O_q(G)) \leqslant N(P)$. By 4.22, we have a contradiction. We conclude that $F(G) = 1$. We have (ii).

Suppose that $C(a) \leqslant N(P)$ for each $a \in P^{\#}$. By 4.3, $G = O_{3'}(G) \cdot N(P)$. But, by (D3) and 4.11, $O_{3'}(G) \leqslant N(P)$. So $G = N(P)$, a contradiction. We conclude that (iii) holds. The lemma is proved.

LEMMA 4.56. *Assume Hypothesis* 4.52. *Suppose that* $1 \neq X \leqslant U$ such that $m(C_W(X)) \geqslant 2$. *Then* $N(X) \leqslant N(P)$.

Proof. Let $N = N(X)$. By definition $X \in \mathscr{X}$ and W_X is defined. Choose W_X in accordance with 4.15. Then $W_X \geqslant C_W(X)$. By 4.14, we may suppose that $N_U(X) \in S_2(N_N(P))$. Assume the notation of 4.13 and set $Q = C_P(X)$.

Then, by 4.13, N, Q, $U \cap N$, V_Y, π satisfy Hypothesis 4.1. Also $N_N(Q) = N_N(P)$. Clearly (D1) and (D4) hold for N. As in 4.54, (D2) holds for N.

Now $V_1 = W_X \geqslant C_W(X)$. Let B be a V_1-invariant 3'-subgroup of N. By 4.11, 4.55(i) and Theorem 6.2.4 of [10], $B \leqslant N(P)$. So $[B, V_1] \leqslant B \cap P = 1$. So $B \leqslant C(V_1)$. We have (D3) for N. We conclude that N satisfies the hypotheses of 4.4.

By 4.55(ii), $N \lneqq G$. So, by minimality of G, $N = N_N(Q) \leqslant N(P)$. The lemma is proved.

LEMMA 4.57. *Assume Hypothesis* 4.52. *Then*

(i) $U \neq 1$.

(ii) $G = O^2(G)$.

(iii) $U \notin S_2(G)$.

(iv) $W \nleqslant C_P(U)$.

(v) $m(V) \geqslant 3$.

Proof. Suppose that $U = 1$. By 4.17(i), $G = O_{3'}(G) \cdot P$. By (D3) and 4.11 $O_{3'}(G) \leqslant N(P)$. So $P \trianglelefteq G$, a contradiction. We conclude that $U \neq 1$ and w have (i).

Now $P \leqslant O^2(G)$. As $P \ntrianglelefteq G$, $P \ntrianglelefteq O^2(G)$. Also $U \cap O^2(G) \in S_2(N(P) \cap O^2(G)$ So, by 4.54, $G = O^2(G)$. We have (ii).

Suppose that $U \in S_2(G)$. By Theorem 7.4.5 (a) of [10] and (D4), as $U \neq$ $O^2(G) \lneqq G$, in contradiction with (ii). So $U \notin S_2(G)$ and we have (iii).

Suppose that $W \leqslant C_P(U)$. By 4.15, we may suppose that $W_X = W$ for eac $X \in \mathscr{X}$. But $W \leqslant N(U)$ and $U \leqslant N(U)$. By 4.55(ii) and 4.54, $N(U) \leqslant N(P)$. E (D1), $U \in S_2(N(U))$ and so $U \in S_2(G)$, in contradiction with (iii). So $W \nleqslant C_P(l$ and we have (iv).

By (iv) and (A4), we have (v) and the lemma.

LEMMA 4.58. *Assume Hypothesis* 4.52. *Then*

(i) *P is not abelian.*

(ii) $d(U) = 1$.

Proof. (i) is immediate from 4.55(i) and (iii).

Suppose that $d(U) \geqslant 2$. By 4.57(iii), $U \notin S_2(G)$. We apply 4.19. By 4.57(v), 4.19 (a) holds. By 4.56, 4.19 (b) holds. By 4.11, 4.55(i) and 4.18, 4.19 (c) holds. By 4.19(iii), there is an involution $x \in U$ such that $C_P(x) = 1$. By Theorem 10.1.4 of [10], P is abelian, in contradiction with (i). So, by 4.57(i), $d(U) = 1$. We have (ii) and the lemma.

Let t denote the unique involution of U. By 4.58(i) and Theorem 10.1.4 of [10], $C_P(t) \neq 1$ and so $\langle t \rangle \in \mathscr{X}$. Set $W^* = W_{\langle t \rangle}$.

LEMMA 4.59. *Assume Hypothesis* 4.52. *Then the following holds.*

(i) *If $a \in W^* - 1$ then $C(a) \leqslant N(P)$.*

(ii) $W^* \leqslant C_P(U)$.

Proof. By 4.58(ii), $t \in Z(U)$. So, by (A3), U normalises W^*. Let $a \in W^* - 1$. In order to prove (i), we may suppose that $C_U(a) \in S_2(C_{N(P)}(a))$. Let $X \in \mathscr{X}$. Then, if $X = 1$, by 4.11, $W \leqslant Z(P) \leqslant C(a)$. If $X \neq 1$ then, by 4.58, $t \in X$. By (A1), $W_X \leqslant C_P(t) \leqslant C(W^*) \leqslant C(a)$. By 4.55(ii), $C(a) \lneqq G$. So, by 4.54, $C(a) \leqslant N(P)$. We have (i).

Suppose that $m(W^*) \geqslant 2$. Let B be a W^*-invariant $3'$ subgroup of G. By (i) and Theorem 6.2.4 of [10], $B \leqslant N(P)$. So $[B, W^*] \leqslant B \cap P = 1$, so that $B \leqslant C(W^*)$. By the same argument as 4.56, we see that $C(t) \leqslant N(P)$.

By 4.58(ii), $N(U) \leqslant C(t) \leqslant N(P)$. But now, as $U \in S_2(N(P))$, $U \in S_2(N(U))$ and so $U \in S_2(G)$. This contradicts 4.57(iii). So $m(W^*) \leqslant 1$. By 4.58(ii), $t \in Z(U)$. So, by (A4), $W^* \leqslant C_P(U)$. We have (iii) and the lemma.

LEMMA 4.60. *Assume Hypothesis* 4.52. *Then $C(a) \leqslant N(P)$ for each $a \in P^{\#}$.*

Proof. For $X \in \mathscr{X}$, define V_X as follows. If $X \neq 1$, let $V_X = W^*$. If $X = 1$, let $V_X = W$. We claim that the hypotheses of 4.4 hold with W_X replaced by V_X. It is sufficient to verigy (A1), (A3) and (A4) for V_X when $1 \neq X \in \mathscr{X}$.

If $1 \neq X \in \mathscr{X}$ then, by 4.58(ii), $t \in X$. So, by 4.59(ii), $W^* \leqslant C_P(X) \leqslant C_P(t)$. By (A1), (A1) holds for V_X.

If $n \in N(P)$ such that $X^n \leqslant U$, by 4.58(ii), $t^n = t$. By (A3), $W^* = W_{\langle t^n \rangle} = (W_{\langle t \rangle})^n = (W^*)^n$. So $V_{X^n} = W^* = (W^*)^n = (V_X)^n$. So (A3) holds for V_X.

If $Y \leqslant N_U(X)$ and $u \in N_U(X) \cap N_U(Y)$ then, by 4.58(ii), $Y \leqslant N_U(\langle t \rangle)$ and $u \in N_U(\langle t \rangle) \cap N_U(Y)$. By (A4), we have (A4) for V_X.

So we can suppose $W_X = W^*$ when $1 \neq X \in \mathscr{X}$. Let $a \in C_P(t)^{\#}$. As U

normalises $C_P(t)$, we can suppose $C_U(a) \in S_2(N_{C(a)}(P))$. Also, by (A1), $W_X \leqslant C(a)$ for each $X \in \mathscr{X}$. By 4.54, $C(a) \leqslant N(P)$. We have proved.

(*) Let $a \in C_P(t)^{\#}$. Then $C(a) \leqslant N(P)$.

Let $b \in P^{\#}$. If $n^{N(P)} \cap C_P(t) \neq \emptyset$ then there is an $n \in N(P)$ such that $b^n \in C_P(t)^{\#}$. By (*), if $m = n^{-1}$, $C(b) = C(b^n)^m \leqslant N(P)$.

So we can suppose $b^{N(P)} \cap C_P(t) = \emptyset$. As $U \in S_2(N(P))$ and, by 4.58(ii), $\langle t \rangle = \Omega_1(U)$, it follows that $C(b) \cap N(P)$ has odd order. By (D2), $C(b) \cap N(P) \leqslant PC(P)$, a 3-nilpotent group. So $C(b) \cap N(P)$ is 3-nilpotent.

Now $W \leqslant C(b)$. Let $Q \in S_3(C(b))$ such that $Q \geqslant W$. By 4.11 and 4.6(i), as $Z(Q) \leqslant Z(J(Q))$, $Q \leqslant P$, $W \leqslant Z(J(Q))$ and $N_{C(b)}(Z(J(Q))) \leqslant C(b) \cap N(P)$, a 3-nilpotent group.

So $N_{C(b)}(Z(J(Q)))$ is 3-nilpotent. By Theorem 8.3.1 of [10], $C(b)$ is 3-nilpotent. But, by (D3) and 4.11, as $W \leqslant C(b)$, $O_{3'}(C(b)) \leqslant N(P)$. So, as $Q \leqslant P \leqslant N(P)$, $C(b) \leqslant N(P)$. The lemma is proved.

Proof of 4.4. 4.60 and 4.55(iii) contradict each other. So no minimal counterexample to 4.4 can exist and so 4.4 is proved.

5. Results on Groups That Admit an Automorphism of Prime Order

We consider groups that satisfy one of the following hypotheses. We label important hypotheses sequentially.

Hypothesis 5.1. G is a group and p is a prime, $p \geqslant 5$. Assume that G admits an automorphism α of order p and set $F = C_G(\alpha)$. Assume that F is a nilpotent p'-group. For any prime q, let $F_q = O_q(F)$.

Let P be an α-invariant Sylow 3-subgroup of G and $K = C_{N(P)}(\alpha)$. For any prime q, let $K_q = O_q(K)$. Assume further that

(A1) For any $q \in \pi(F)$ there is one and only one α-invariant Sylow q-subgroup of G.

Hypothesis 5.2. Assume Hypothesis 5.1. Assume further that there is a subgroup $W \leqslant \Omega_1(Z(P))$ such that W is α-invariant and

(A2) $N(P) = N(W) = PC(P)K$.

(A3) W is weakly closed in P with respect to G.

(A4) Either $C_W(\alpha) = 1$ or $W \leqslant F$.

In Section 6 we shall require the following results.

Theorem 5.3. *Assume Hypothesis* 5.2. *Assume further that* $K_2 \leqslant C(P)$. *Then* $G = O_{3'}(G) \cdot N(P)$.

THEOREM 5.4. *Assume Hypothesis* 5.1. *Let H be an α-invariant normal subgroup of G such that $P \leqslant H$. Let W be a non-identity α-invariant subgroup of $Z(P)$ such that $C_W(\alpha) = 1$. Assume further that*

(a) $N(W) = N(P) = N(Z(J(P))) = PC(P)K$.

(b) $C_H(W) = PC_H(P)$.

Let $q \in \pi(H) - \{3\}$ and $И_H^*(W; q)$ be the set of maximal W-invariant q-subgroups of H. Then the following holds.

(1) *The elements of $И_H^*(W; q)$ are permuted transitively by $C_H(W)$.*

(2) *There is a $Q \in И_H^*(W; q)$ such that Q is α-invariant and $C_Q(W) \in S_q(C_H(W))$.*

(3) $N(P) = C_H(W) \cdot (N(P) \cap N(Q))$, *for Q as in* (2).

(4) *For Q as in* (2), $N(P) \cap N(Q)$ *contains an α-invariant Sylow q-subgroup of $N(P)$.*

THEOREM 5.5. *Assume Hypothesis* 5.1. *Assume further that*

(a) $F_3 = 1$.

(b) $N(P) = F(N(P))K$.

(c) *Whenever H is an α-invariant solvable subgroup of G such that $Z(K) \leqslant H$ and $P \cap H \in S_3(H)$, then $P \cap H \trianglelefteq H$.*

Assume further that if G does not satisfy Hypothesis 5.2, *then there is a non-empty set π^* of primes, $\pi^* \subseteq \{2, 3\}'$, such that whenever $r \in \pi^*$ and R is an α-invariant Sylow r-subgroup of G, then for some subgroup W_R of $Z(R) \cap K$,*

(d) $N(W_R) = N(R) = RF$.

(e) *W_R is weakly closed in R with respect to G.*

(f) *If $t \in \pi(N(P)/F(N(P))) - \{2\}$ such that $[O_t(N(P)), \alpha] \neq 1$, then $t \in \pi^*$.*

Then $P \trianglelefteq G$.

THEOREM 5.6. *Assume Hypothesis* 5.2. *Assume further that $F = K$. Then $[P, \alpha] \trianglelefteq G$.*

THEOREM 5.7. *Assume Hypothesis* 5.2. *Assume further that*

(a) $F_3 = 1$.

(b) *Each W-invariant $3'$-subgroup of G lies in $C(W)$.*

(c) $O(K) \leqslant C(P)$.

(d) $K_2 \in S_2(N(P))$.

Then $P \trianglelefteq G$.

5.3 is proved by a method that appears in our thesis [23]. The proof of 5.4 is an adaptation of the proof of Theorem 3.1 of [13]. The proofs of 5.5 and 5.6 will be discussed more fully below.

In bief, we show that a minimal counterexample G to 5.5 or 5.6 satisfies the following hypothesis.

HYPOTHESIS 5.8. (B1) G satisfies Hypothesis 5.1.

(B2) G satisfies the hypotheses of 5.5 or the hypotheses of 5.6.

(B3) Whenever H is an α-invariant proper subgroup of G such that $Z(K) \leqslant H$ and $P \cap H \in S_3(H)$, then H is solvable and $[P \cap H, \alpha] \trianglelefteq H$.

(B4) $F(G) = 1$.

(B5) $P = [P, \alpha] \neq 1$.

The main results of this section is that no group can satisfy Hypothesis 5.8. This is Proposition 5.14. The proof is relatively long and will be discussed fully below.

LEMMA 5.9. *Let G be a group and p be a prime. Assume that G admits an automorphism α of order p such that $C_G(\alpha)$ is a p'-group. Assume further that G is non-solvable, but that every α-invariant proper subgroup of G is solvable. Finally assume that, whenever H is a non-identity α-invariant normal subgroup of G, G/H is solvable. Then G is simple.*

Proof. Clearly G is characteristically simple. So, by Theorem 2.1.4 of [10], $G = G_1 \times G_2 \times \cdots \times G_n$, where the G_i are isomorphic simple groups. Now α permutes the G_i. By the condition on α-invariant proper subgroups of G, α acts transitively on the G_i. Clearly we may suppose $n = p$.

Let $F = \{xx^{\alpha} \cdots x^{\alpha^{p-1}}: x \in G_1\}$. Then $G_1 \cong F \leqslant C_G(\alpha)$. But $C_G(\alpha)$ is solvable, a contradiction. The lemma follows.

LEMMA 5.10. *Let G be a group and p be a prime. Assume that G admits an automorphism α of order p such that $C_G(\alpha)$ is a nilpotent p'-group. Assume further that either G is S_4-free or G has no elements of order 6. Then G is solvable.*

Proof. Assume false and let G denote a minimal counterexample. By an easy argument, using 5.9, G is simple. Using Theorem C of [7] and the main theorem of [25], we see that no counterexample to the lemma appears on their lists. The lemma is proved.

The next lemma is an easy corollary of Theorem 6.2.2 of [10].

LEMMA 5.11. *Let G be a group and assume that G admits an automorphism α of order p such that $F = C_G(\alpha)$ is a nilpotent p'-group. Let $q \in \pi(G)$ and Q be an*

α-invariant Sylow q-subgroup of G. Then $Q \geqslant O_q(F)$. Furthermore, if $F \leqslant N(Q)$, then the following holds.

(i) *Q is the only α-invariant Sylow q-subgroup of G.*

(ii) *If X is an α-invariant q-subgroup of G then $X \leqslant Q$.*

(iii) *If H is an α-invariant subgroup of G then $H \cap Q \in S_q(H)$.*

LEMMA 5.12. *Let G be a solvable group that satisfies Hypothesis 5.2. Then $G = O_{3'}(G) \cdot N(P)$.*

Proof. By Theorem 6.3.3 of [10], $W \leqslant O_{3',3}(G)$. Let $Q = P \cap O_{3',3}(G)$. Then $Q \in S_3(O_{3',3}(G))$. As $W \leqslant Q \leqslant P$, by (A2), (A3) and 4.6(i), $N(Q) \leqslant N(P)$. So, by the Frattini argument, $G = O_{3',3}(G) \cdot N(Q) = O_{3'}(G) \cdot N(Q) = O_{3'}(G) \cdot N(P)$. The lemma follows.

We now prove 5.3 and 5.4.

Proof of 5.3. As $K_2 \leqslant C(P)$, $|N(P) : C(P)|$ is odd. Let $P_1 \leqslant P$. Then

$$C(P) \leqslant C(P_1). \tag{1}$$

Let $P_2 \in S_3(C(P_1))$, $P_2 \geqslant W$. By 4.6(i), $N(P_2) \leqslant N(W) = N(P)$. By the Frattini argument,

$$N(P_1) = C(P_1) \cdot (N(P_1) \cap N(P)). \tag{2}$$

By (1) and (2), $|N(P_1) : C(P_1)|$ is odd. By Sylow's theorem we have that if X is a 3-subgroup of G then $\mathrm{Aut}_G(X)$ has odd order.

Assume that D_6 is involved in G. Let $H, K \leqslant G$ such that $K \trianglelefteq H$ and $H/K \cong D_6$. Let $L = O_3(H \bmod. K)$ and $Y \in S_3(L)$. Then, by the Frattini argument, $H = LN_H(Y) = KYN_H(Y) = KN_H(Y)$. So $N_H(Y)/N_L(Y) \cong D_6$ and $2 \in \pi(\mathrm{Aut}_G(Y))$, a contradiction. So D_6 and hence S_4 is not involved in G. By 5.10 and 5.12, $G = O_{3'}(G) \cdot N(P)$.

Before proving 5.4, we require a preliminary lemma which will be useful later on as well.

LEMMA 5.13. *Assume that G satisfies Hypothesis 5.1. Assume that $N(P) = (PC(P))K$. Let H be an α-invariant subgroup of G such that $Q = P \cap H$ and $N_H(Q) \leqslant N(P)$. Then $N_H(Q) = (QC_H(Q))(K \cap H)$ and $Q \in S_3(H)$.*

Proof. As $N_H(Q) \leqslant N(P)$, $N_H(Q)$ is 3-closed. So $Q = P \cap H \in S_3(N_H(Q))$, so that $Q \in S_3(H)$.

Let $N = N_H(Q)$. Then $[N, \alpha] \leqslant PC(P) = P \times O_{3'}(N(P))$. So $[N, \alpha] \leqslant Q \times O_{3'}(N)$. By Theorem 6.2.2 of [10], $N = [N, \alpha] \cdot C_N(\alpha) = (Q \times O_{3'}(N)) \cdot (K \cap N)$. The lemma follows.

Proof of 5.4. By Hypothesis 5.1, $p \geqslant 5$. As $C_W(\alpha) = 1$ and $1 \neq W = W^\alpha \leqslant Z(P)$, $m(W) \geqslant 3$. Let $V \leqslant W$ such that $m(W/V) \leqslant 2$. We show firstly that $C_H(V)$ is 3-nilpotent.

By 5.13, as $P \leqslant H$,

$$N_H(P) = (PC_H(P))(K \cap H). \tag{3}$$

Set $L = K \cap H$ and $C = C_H(V) \cap N_H(P)$. Then $PC_H(P) \leqslant C$. So, by (3),

$$C = PC_H(P)(L \cap C).$$

Let $x \in (L \cap C) - C_H(W)$. Then, as $C_W(\alpha) = 1$, $m([W, x]) \geqslant 3$. But $V \leqslant C_W(x)$. By Theorem 5.2.3 of [10], as $P \leqslant C(V)$, we have contradicted the assumption $m(W/V) \leqslant 2$. So $L \cap C \leqslant C_H(W)$ and so, by (b),

$$C = PC_H(P) = C_H(W) = P \times O_{3'}(N(P)). \tag{4}$$

Thus C is 3-nilpotent. Let $D = C_H(V)$. Then $P \leqslant D$. As $N(P) = N(Z(J(P)))$,

$$N_D(Z(J(P))) = N_D(P) = C_H(V) \cap N_H(P) = C.$$

So $N_D(Z(J(P)))$ is 3-nilpotent. By Theorem 8.3.1 of [10], $D = C_H(V)$ is a 3-nilpotent group, as claimed.

Let $\mathscr{C}_1, \ldots, \mathscr{C}_n$ be the conjugate classes of $\mathsf{И}_H^*(W; q)$, under the action of $C_H(W)$. We now show that $n = 1$.

Suppose that $n \geqslant 2$. Among all possible choices, let $1 \leqslant i, j \leqslant n$, $Q_1 \in \mathscr{C}_i$ and $Q_2 \in \mathscr{C}_j$ such that $|Q_1 \cap Q_2|$ is maximal. Set $E = Q_1 \cap Q_2$ and $S_k = N_{Q_k}(E)$. Then W acts on S_1/E and S_2/E. So there are subgroups V_1, V_2 of W such that $|W : V_i| \leqslant 3$, $1 \leqslant i \leqslant 2$ and $C_{S_k/E}(V_k) \neq 1$, $1 \leqslant k \leqslant 2$.

Set $V = V_1 \cap V_2$. Then $m(W/V) \leqslant 2$ and so $C_H(V)$ is 3-nilpotent. Set $R_k = C_{S_k}(V)$, $1 \leqslant k \leqslant 2$. Then $R_k \nleqslant E$, $1 \leqslant k \leqslant 2$. As $C_H(V)$ is 3-nilpotent,

$$R_k \leqslant O_{3'}(C_H(V) \cap N_H(E)), \qquad 1 \leqslant k \leqslant 2. \tag{5}$$

Recall $E = Q_1 \cap Q_2$. Set $M = E \cdot (O_{3'}(C_H(V)) \cap N_H(E))$. Then $R_k \leqslant M$, $1 \leqslant k \leqslant 2$. Clearly M is a W-invariant $3'$-subgroup of H. Let $T_k = ER_k$, $1 \leqslant k \leqslant 2$. By Theorem 6.2.2 of [10], there is an $x \in C_H(W)$ such that $\langle T_1, T_2^x \rangle$ is a W-invariant q-subgroup of H. Let $Q^* \in \mathsf{И}_H^*(W; q)$, $Q^* \geqslant \langle T_1, T_2^x \rangle$.

Then, as $R_k \nleqslant E$, $1 \leqslant k \leqslant 2$, $|Q^* \cap Q_1| > |E|$ and $|Q^{*x^{-1}} \cap Q_2| > |E|$. As $x \in C_H(W)$, by maximality of $|E|$, $Q^* \in \mathscr{C}_i \cap \mathscr{C}_j$, a contradiction. So $n = 1$ and we have (1).

It follows from (1) that $|\mathsf{И}_H^*(W; q)|$ divides $|C_H(W)|$. By 2.3(i), $p \nmid |G|$ and so $p \nmid |\mathsf{И}_H^*(W; q)|$. As $W = W^\alpha$ and $H = H^\alpha$, α permutes the members of

$\mathit{V}^*_H(W; q)$. So there is a $Q \in \mathit{V}^*_H(W; q)$ such that $Q = Q^\alpha$. Let $Q_0 \in S_q(C_H(W))$. Then there is an $x \in C_H(W)$ such that $Q_0{}^x \leqslant Q$. But $Q_0{}^x \in S_q(C_H(W))$. So $C_Q(W) \in S_q(C_H(W))$. We have (2).

As $H \trianglelefteq G$ and $N(P) = N(W)$, $N(P)$ permutes the members of $\mathit{V}^*_H(W; q)$by conjugation. Let $n \in N(P)$. Then $Q^n \in \mathit{V}^*_H(W; q)$. So there is a $d \in C_H(W)$ such that $Q^n = Q^d$. So $nd^{-1} \in N(Q) \cap N(P)$. We conclude that $N(P) = C_H(W) \cdot (N(P) \cap N(Q))$ and we have (3).

By 5.4(2), 5.4(3) and an easy counting argument, we have (4). The result is proved.

We now show that no group can satisfy Hypothesis 5.8.

PROPOSITION 5.14. *No group can satisfy Hypothesis* 5.8.

Until we prove 5.14, suppose not and let G satisfy Hypothesis 5.8. Let $\mathscr{H}$ denote the set of α-invariant normal subgroups H of G such that $P \leqslant H$. For $H \in \mathscr{H}$, let σ_H denote the set of $r \in \pi(\mathrm{Aut}_H(P)) - \{3\}$ such that, for some $x \in (K_r \cap H) - C_H(P)$, $[C_P(x), \alpha] \neq 1$. Note that, by (B2), $N(P) = PC(P)K$. Also $\sigma_H \subseteq \pi(K)$.

It is easy to see that if $H, L \in \mathscr{H}$ such that $H \leqslant L$ then $\sigma_H \subseteq \sigma_L$. We show first that $\sigma_H = \emptyset$ for some $H \in \mathscr{H}$. We then apply 4.2 and 4.3 to such an H in order to obtain the desired contradiction. Our first task is to show that G satisfies Hypothesis 5.2.

LEMMA 5.15. *Assume that G satisfies the hypotheses of* 5.5. *Then, there is a $W \leqslant \Omega_1(Z(P))$ such that W is weakly closed in P with respect to G and $N(W) = N(P)$.*

Proof. We require the following. (Note that $F_3 = C_P(\alpha) = 1$.)

(*) *Let P_0 be a non-identity $Z(K) \cdot \langle\alpha\rangle$-invariant subgroup of P. Then $N(P_0) \leqslant N(P)$.*

Assume that (*) is false. Let P_0 be maximal subject to violating (*). Let $P_1 = N_P(P_0)$. Then P_1 is a non-identity $Z(K) \cdot \langle\alpha\rangle$-invariant subgroup of P. Clearly $P_1 > P_0$. So, by maximality of P_0, $N(P_1) \leqslant N(P)$. We conclude that $P_1 \in S_3(N(P_0))$. But now, by (B3) and (B4), as $Z(K) \leqslant N(P_0)$, $N(P_0) \leqslant N(P_1) \leqslant N(P)$, a contradiction. We have proved (*).

We can suppose that 5.5(d), (e) and (f) apply. Set $W = \Omega_1(Z(P))$. By (*), $N(W) = N(P) = N(J(P))$. So

$$W \trianglelefteq N(J(P)). \tag{6}$$

By 5.5 (b), $N(W) = C(W)K$. Let $x \in N(W) - C(W)$. Then $x = yz$ where $y \in C(W)$ and $z \in K$. So $W \geqslant [W, x] = [W, z]$, an α-invariant subgroup of G. By Hypothesis 5.1, $p \geqslant 5$. By 5.5 (a), as W is elementary abelian,

$$m([W, x]) = m([W, z]) \geqslant 3. \tag{7}$$

By 3.4, we may suppose that there is an $r \in \pi(\mathrm{Aut}_G(W)) - \{2\}$, an r-subgroup R of G and a 2-element t of G such that

(i) $O_r(N(W)) \leqslant R \leqslant N(W)$.

(ii) $t^2 \in N(W) - C(W)$.

(iii) $t \in N(R)$.

(iv) $[R, t] \nleqslant C(W)$.

Let $R^{*\alpha} = R^* \in S_r(N(P))$. As $N(W) = N(P)$, we may suppose that $R \leqslant R^*$. By 5.5 (b),

$$[R, \alpha] \leqslant [R^*, \alpha] \leqslant O_r(N(W)) \leqslant R. \tag{8}$$

So R is α-invariant. Let $S^\alpha = S \in S_r(G)$ such that $S \geqslant R^*$.

If $R \leqslant F$ then, by 2.2, as F is nilpotent, $N(R)/C(R)$ is an r-group. This contradicts (iii) and (iv). So $R \nleqslant F$. By (8), as $N(W) = N(P)$

$$[O_r(N(P)), \alpha] \neq 1. \tag{9}$$

By (iii) and (iv), $R \nleqslant C(W)$. So

$$r \in \pi(N(P)/F(N(P))) - \{2, 3\}. \tag{10}$$

By 5.5 (f), $r \in \pi^*$.

Let $X = [R, t^2]$ and suppose that $X \neq 1$. We argue that $N(X) \leqslant N(P)$. Let $T^\alpha = T \in S_2(N(P))$. Then there is an $n \in N(P)$ such that $(t^2)^n \in T$. So, if $u = (t^2)^n$, by 5.5 (b), $u = u_1u_2$ where $u_1 \in O_2(N(P))$ and $u_2 \in K_2$. Now, by 5.5 (b), $N(P)/F(N(P))$ is nilpotent. So, by Theorem 5.3.6 of [10] and (i), $[R, t^2] \leqslant [O_r(N(P)), t^2] \leqslant [R, t^2]$. So $X = [O_r(N(P)), t^2]$ and $X^n = [O_r(N(P)), u] = [O_r(N(P)), u_2]$ is α-invariant.

Since $X \leqslant O_r(N(P))$, $P \leqslant N(X)$. By (B3) and (B4), as $X^n \leqslant O_r(N(P))$, $N(X^n) \leqslant N(P)$. As $N(X^n) = N(X)^n$, $N(X) \leqslant N(P)$. By (iii), $t \in N(X) \leqslant N(P)$. But $N(P)/F(N(P))$ is nilpotent. So $[R, t] \leqslant F(N(P)) \leqslant C(W)$, in contradiction with (iv). So $X = 1$.

Recall that there is an $n \in N(P)$ such that $u = (t^2)^n \in T$. By 5.5 (b), $u = u_1u_2$ where $u_1 \in O_2(N(P))$ and $u_2 \in K_2$. Let v be the unique involution of $\langle u_2 \rangle$. As $X = 1$, $[R^n, u] = 1$. So, by (i), $[O_r(N(P)), u] = 1$. So $[O_r(N(P)), v] = 1$. Recall that $R^* = R^{*\alpha} \in S_r(N(P))$. By 5.5 (b), as F is nilpotent, $R^* = O_r(N(P)) \cdot K_r \leqslant C(v)$.

Also, by (B3), (B4) and (9),

$$N(P) = N(O_r(N(P))). \tag{11}$$

Recall $S^\alpha = S \in S_r(G)$, $S \geqslant R^*$. Then $C_S(R^*) \leqslant C_S(O_r(N(P))) \leqslant N(P) \cap S = R^*$. By (10) and 5.5 (b), $r \in \pi(F)$. So, by (A1), $v \in F \leqslant N(S)$. We deduce from Theorem 5.3.4 of [10] that

$$S \leqslant C(v). \tag{12}$$

Clearly we can suppose that P is not abelian. By Theorem 10.1.4 of [10], $C_P(v) \neq 1$. Set $C = C(v)$ and $P_0 = C_P(v)$. Then P_0 is $Z(K)\langle\alpha\rangle$-invariant. By (*),

$$N_C(P_0) \leqslant N(P) \cap C.$$

Thus $P_0 \in S_3(N_C(P_0))$ and so $P_0 \in S_3(C)$. By (B3) and (B4), $P_0 \trianglelefteq C$. So

$$C \leqslant N(P). \tag{13}$$

But now, by (12), $S \leqslant N(P)$. By 5.5 (b), $[S, \alpha] \leqslant O_r(N(P))$. By (B3), (B4) and (9), $N([S, \alpha]) \leqslant N(P)$. As $F \leqslant N([S, \alpha])$, $F \leqslant N(P)$. We deduce from 5.5 (d) that $N(W_S) = N(S) \leqslant N(P)$.

Now $W_S \leqslant C(R)$. By (11) and (i), $C(R) \leqslant N(P)$. Let $R_0 \in S_r(C(R))$, $R_0 \geqslant W_S$. By 4.6(i), 5.5 (d) and 5.5 (e),

$$N(R_0) \leqslant N(W_S) = N(S) \leqslant N(P). \tag{14}$$

By (14) and the Frattini argument, $N(R) = C(R)(N(R_0) \cap N(R)) \leqslant N(P)$. So, by (iii), $t \in N(P)$. But, by 5.5 (b), $N(P)/F(N(P))$ is nilpotent. So

$$[R, t] \leqslant F(N(P)) \leqslant C(W),$$

in contradiction with (iv). We conclude that W is weakly closed in P with respect to G. The lemma is proved.

Corollary 5.16. *G satisfies Hypothesis* 5.2.

Proof. Immediate from (B1), (B2) and 5.15.

From now on let W denote an α-invariant subgroup of $\Omega_1(Z(P))$ such that G, P, W satisfy Hypothesis 5.2.

Lemma 5.17. *Assume that $W \nleqslant F$. Then, if P_0 is a non-identity $Z(K)\langle\alpha\rangle$-invariant subgroup of P, $N(P_0) \leqslant N(P)$.*

Proof. Now $W \leqslant N(P_0)$. Let $P_1 \in S_3(N(P_0))$, $P_1 \geqslant W$. By 5.16 and 4.6(i), $P_1 \leqslant P$. So $P \cap N(P_0) \in S_3(N(P_0))$. By (B3) and (B4), $[N_P(P_0), \alpha] \trianglelefteq N(P_0)$. But, by (A4), $W \leqslant [N_P(P_0), \alpha]$. Thus, by 5.16 and 4.6(i), $N(P_0) \leqslant N(P)$. The lemma is proved.

Lemma 5.18. *Let $\pi = \pi(N(P)/PC(P))$, U be an α-invariant Hall π-subgroup of $N(P)$, $\mathscr{X} = \{X \leqslant U\colon C_P(X) \neq 1\}$ and, for $X \in \mathscr{X}$, let $W_X = \Omega_1(Z(C_P(X)))$. Then G, P, π, U and W_X satisfy Hypothesis* 4.1. *Furthermore $2 \in \pi$ and $N(P) = PC(P)U$.*

Proof. By (B3), (B4) and (B5), $N(P)$ is solvable. So U exists. By (B2), $N(P) = PC(P)K$. By (B3) and (B4), if $Z = \Omega_1(Z(P))$, $N(P) = N(Z)$. We have 4.8 (a) and 4.8 (b). By 5.16, $W \leqslant Z$. If $g \in G$ such that $Z^g \leqslant P$ then $W^g \leqslant P$. So, by 5.16, $g \in N(P)$ and $Z^g = Z$. So 4.8 (c) holds. In order to verify 4.8 (d), by 5.17, we may suppose $W \leqslant F$. By 5.16, $W \leqslant C_Z(\alpha)$. Let Q be a non-identity $Z(K)\langle\alpha\rangle$-invariant subgroup of P such that $Q \geqslant C_Z(\alpha)$. Then $Q \geqslant W$ and, by 5.16 and 4.16(i), $N(Q) \leqslant N(P)$. We have 4.8 (d).

By 4.8, we just have to show that $2 \in \pi$. If $2 \notin \pi$ then, as $N(P) = PC(P)K$, $K_2 \leqslant C(P)$. By 5.3, $G = O_{3'}(G) \cdot N(P)$. Now, by 5.10, $O_{3'}(G)$ is solvable. So, by (B4), $O_{3'}(G) = 1$ and $P \trianglelefteq G$. By (B4) and (B5), we have a contradiction. So $2 \in \pi$ and the lemma is proved.

LEMMA 5.19. *Let $q \in \pi(K) - \{3\}$ and $x \in K_q^\#$ such that $C_P(x) \neq 1$. Then $K = F$.*

Proof. By (A1), we may suppose $F_3 = 1$. By 5.17, $N(C_P(x)) \leqslant N(P)$. So $C_P(x) \in S_3(N_{C(x)}(C_P(x)))$ and so $C_P(x) \in S_3(C(x))$. We conclude from (B3) and (B4), that

$$C(K_q) \leqslant C(x) \leqslant N(C_P(x)) \leqslant N(P). \tag{15}$$

Thus $C(K_q) \leqslant N(P)$. Let $Q^\alpha = Q \in S_q(G)$. If $Q = F_q$ then, by 2.6, G is q-nilpotent. So, by Theorem 6.2.2 of [10], $F_q \leqslant N(P)$. As $O_{q'}(F) \leqslant C(K_q) \leqslant N(P)$, $F \leqslant N(P)$.

So we may suppose that $Q > F_q$. By Theorem 5.3.4 of [10], $Z = [C_Q(F_q), \alpha] \neq 1$. By (A1), $F \leqslant N(Z)$. By (B3), (B4) and 2.14, $Z \leqslant O_q(C(K_q))$.

Let $R = N_{F_q}(K_q)$. Then R normalizes $[O_q(C(K_q)), \alpha] = T$ (say). By (15), $T \leqslant N(P)$. By 2.14, (B3), (B4) and (B5), $T \leqslant O_q(N(P))$ and so $P \leqslant N(T)$. As $Z \leqslant O_q(C(K_q))$, by Theorem 5.3.6 of [10], $1 \neq Z \leqslant T$. So, by (B3) and (B4), $N(T) \leqslant N(P)$. Thus $R \leqslant N(P)$. But now $R = K_q$. So $F_q = K_q$. As, by (15), $O_{q'}(F) \leqslant C(K_q) \leqslant N(P)$, $F \leqslant N(P)$. The lemma is proved.

LEMMA 5.20. *Let $q \in \pi(K) - \{2\}$ such that $d(K_q) \geqslant 2$. Let Q be an α-invariant Sylow q-subgroup of G and assume that $[Q, \alpha] \neq 1$. Let $Y = \langle O_{q'}(C(x)) \colon x \in F_q^\# \rangle$. Then*

(i) *$K = F$.*

(ii) *Y is a solvable q'-subgroup of G.*

(iii) *Let $A \leqslant F_q$ such that $d(A) \geqslant 2$ and $g \in G$ such that $A^g \leqslant N(Y)$, then $g \in N(Y)$.*

(iv) *If $q \neq 3$, then $P \leqslant N(Y)$.*

(v) *If $Y \neq 1$ then $Y = O_{q'}(N(P))$ and $N(Y) = N(P)$.*

Also $N(Q) \leqslant N(P)$.

Proof. By Theorem 6.2.4 of [10], $C_P(x) \neq 1$ for some $x \in K_q^{\#}$. So, by (A1 and 5.19, we have (i).

Let $Z = [C_Q(F_q), \alpha]$. Now $q \in \pi(F)$. By (A1) and 5.11, $F_q \leqslant Q$. By Theorem 5.3.4 of [10], $Z \neq 1$. Also, by (A1), $F \leqslant N(Z)$. Let $x \in F_q^{\#}$. By (i) and Theorem 6.2.2 of [10], $C_P(x) \in S_3(C(x))$. By (i), $Z(K) \leqslant C(x)$. So, by (B3) and (B4), $C(x)$ is solvable. Also $Z \leqslant C(x)$ and $C_{C(x)}(\alpha) \leqslant N(Z)$. By 2.14, $Z \leqslant O_q(C(x))$. Thus $O_{q'}(C(x)) \leqslant C(Z)$ for each $x \in F_q^{\#}$.

By (i) and Theorem 6.2.2 of [10], $N_P(Z) \in S_3(N(Z))$. Also $Z(K) \leqslant F \leqslant N(Z)$ So, by (B3) and (B4), as $Z \neq 1$, $N(Z)$ is solvable. By 4.5, $O_{q'}(C(x)) \leqslant O_{q'}(N(Z))$ for each $x \in F_q^{\#}$. Thus $Y \leqslant O_{q'}(N(Z))$ and so we have (ii).

Let $A \leqslant F_q$ such that $d(A) \geqslant 2$ and $g \in G$ such that $A^g \leqslant N(Y)$. Then, by Theorem 6.2.4 of [10],

$$Y = \langle C_Y(a^g) \colon a \in A^{\#} \rangle. \tag{16}$$

Let $a \in A^{\#}$. We claim that $C_Y(a^g) \leqslant O_{q'}(C(a^g))$. Clearly we may suppose $Y \neq 1$. By (B4) and (ii), $N(Y) < G$. By (i) and 5.11, $N_P(Y) \in S_3(N(Y))$. Clearly $Z(K) \leqslant F \leqslant N(Y)$. So, by (B3), $N(Y)$ is solvable. Thus, by 2.14, $Z \leqslant O_q(N(Y))$.

Ley $Q_0^{\alpha} = Q_0 \in S_q(N(Y))$ and $X = Z(Q_0) \cap O_q(N(Y))$. Then $X \neq 1$. By (A1) and Theorem 6.2.2 of [10], $F \leqslant N(Q_0)$. Thus $Z(K) \leqslant N(X)$. As before $N(X)$ is solvable. By 4.5, as $Z \leqslant N(X)$,

$$Y \leqslant O_{q'}(N(Z)) \cap N(X) \leqslant O_{q'}(N(X)).$$

Now there is an $n \in N(Y)$ such that $A^g \leqslant Q_0^n$. Thus $X^n \leqslant C(a^g)$. As before $C(a)$ is solvable. But $C(a^g) = C(a)^g$. So $C(a^g)$ is solvable. Thus, by 4.5,

$$C_Y(a^g) \leqslant O_{q'}(N(X^n)) \cap C(a^g) \leqslant O_{q'}(C(a^g)) = O_{q'}(C(a))^g \leqslant Y^g.$$

By (16), $Y \leqslant Y^g$ and so $g \in N(Y)$. We have (iii).

Suppose that $q \neq 3$. By 4.7 and (B5),

$$P = \langle [C_P(x), \alpha] \colon x \in K_q^{\#} \rangle.$$

Let $x \in K_q^{\#}$. Recall that $C(x)$ is solvable. Then, by (i) and 2.14, $[C_P(x), \alpha] \leqslant O_3(C(x)) \leqslant O_{q'}(C(x)) \leqslant Y$. Thus $P \leqslant Y$ and we have (iv).

Suppose that $Y \neq 1$. We note that, by (iii), $N(Q_0) \leqslant N(Y)$. So $Q_0 \in S_q(N(Q_0))$ and $Q_0 \in S_q(G)$. By (A1) and 5.11, $Q_0 = Q$ and $N(Q) \leqslant N(Y)$.

If $q = 3$ then $P \leqslant N(Y)$. By (iv), $P \leqslant N(Y)$ for any q. So, by (B3) and (B4), as $F \leqslant N(Y)$, $N(Y) \leqslant N(P)$.

By 4.5 and (i), $Y \leqslant O_{q'}(N(P)) = L$ (say). Also $Q \leqslant N(Y) \leqslant N(P)$. Now, by 2.14, as $N(P)$ is solvable, $Z \leqslant O_q(N(P))$. So $X^* = Z(Q) \cap O_q(N(P)) \neq 1$.

By (A1) and (i), $F \leqslant N(X^*)$. So, by (B3) and (B4), $N(X^*) = N(P)$. Thus $L = O_{q'}(N(X^*))$. For any $x \in F_q^{\#}$, $C_L(x) \leqslant O_{q'}(N(X^*)) \cap C(x)$. By 4.5, as

$X^* \leqslant C(Q) \leqslant C(x)$, $C_L(x) \leqslant O_{q'}(C(x)) \leqslant Y$. By Theorem 6.2.4 of [10], $L \leqslant Y$. Thus $Y = L = O_{q'}(N(P))$.

As $N(Y) \leqslant N(P)$, $N(Y) = N(P)$. We have (v) and the lemma.

LEMMA 5.21. *Let $q \in \pi(F) - \{2, 3\}$ and $H = O^q(G)$. Let Q be an α-invariant Sylow q-subgroup of G. Assume that $N(Q) \leqslant N(P)$. Then $q \notin \pi(\mathrm{Aut}_H(P))$ and $q \notin \sigma_H$.*

Proof. Let $R = Q \cap H$. Then $R \in S_q(H)$ and

$$Q \leqslant N(R). \tag{17}$$

Set $N = N(K_\infty(R))$. We apply Glauberman's p-factor theorem [9]. Clearly we may suppose $R \neq 1$.

By (A1), $F \leqslant N(Q) \leqslant N$. As $N(Q) \leqslant N(P)$, $F \leqslant N(P)$. So, by 5.11, $N \cap P \in S_3(N)$. By (B3) and (B4), N is solvable. Set $Q^* = [Q, \alpha]$. By 2.14,

$$Q^* \leqslant O_q(N). \tag{18}$$

By solvability of $N(P)$ and 2.14, $Q^* \leqslant O_q(N(P))$. As $R \neq 1$, by 2.6, $Q^* \neq 1$. Thus, by (B3) and (B4), $P \trianglelefteq N(Q^*)$ and so $N(Q^*) \leqslant N(P)$. By (18),

$$O_{q'}(N) \leqslant N(Q^*) \leqslant N(P). \tag{19}$$

Suppose that $d(F_q) \geqslant 2$. As $F \leqslant N(P)$, $d(K_q) \geqslant 2$. If $d(C_{O_q(N)}(\alpha)) \geqslant 2$, then, by 5.20(iii), (iv) and (v), $N \leqslant N(O_q(N)) \leqslant N(P)$. If $d(C_{O_q(N)}(\alpha)) \leqslant 1$ then, by 2.16, as q is odd, N has q-length 1. Similarly, if F_q is cyclic then, by 2.13, N has q-length 1. So, by (17), $Q \leqslant O_{q',q}(N)$. By the Frattini argument, (19) and the assumption that $N(Q) \leqslant N(P)$,

$$N = O_{q',q}(N)\, N_N(Q) = O_{q'}(N)\, N_N(Q) \leqslant N(P).$$

So, in any case, $N \leqslant N(P)$. By Glauberman's p-factor theorem [9], $R \leqslant N \cap H = O^q(N \cap H) \leqslant O^q(N(P))$. By (B2), $N(P)/PC(P)$ is nilpotent. So $R \leqslant PC(P)$ and so $R \leqslant C_H(P)$. But now $q \notin \pi(\mathrm{Aut}_H(P))$ and so $q \notin \sigma_H$. The lemma is proved.

If $\sigma_G \neq \varnothing$, set $H = \bigcap_{q \in \sigma_G} O^q(G)$. If $\sigma_G = \varnothing$, set $H = G$. We will now show that $\sigma_H = \varnothing$. Clearly $H \in \mathscr{H}$.

COROLLARY 5.22. *Let $q \in \sigma_H - \{2\}$. Then $d(K_q) = 1$.*

Proof. Now $q \in \pi(F) - \{2, 3\}$. Suppose that $d(K_q) \geqslant 2$. By 2.6, $F_q \notin S_q(G)$. Thus, by 5.20 and (B5), if $Q^\alpha = Q \in S_q(G)$, $N(Q) \leqslant N(P)$. So, by 5.21, if $L = O^q(G)$, $q \notin \sigma_L$. But $H \leqslant L$. As pointed out after the statement of 5.14, this means that $\sigma_H \subseteq \sigma_L$, a contradiction. We conclude that $d(K_q) = 1$, proving the corollary.

LEMMA 5.23. *Assume that $W \nleqslant F$. Then $\sigma_H \subseteq \{2\}$.*

Proof. By (B2), $N(P) = PC(P)K$. Let $q \in \sigma_H - \{2\}$ and $Q^\alpha = Q \in S_q(N(P))$. By (B2), $Q = C_Q(P)K_q$. By definition of σ_H, $q \in \sigma_G$ and there is an $x \in K_q - C_Q(P)$ such that $[C_P(x), \alpha] \neq 1$. By 5.22, $Q/C_Q(P)$ is cyclic.

Now $\Omega_1(Q)\, C_Q(P)/C_Q(P) \leqslant \Omega_1(Q/C_Q(P))$. So

$$1 \neq [C_P(x), \alpha] \leqslant [C_P(\Omega_1(Q)), \alpha]. \tag{20}$$

As $q \in \sigma_H$, by 5.19, $K = F$. By Theorem 6.2.2 of [10], $N_P(\Omega_1(Q)) \in S_3(N(\Omega_1(Q)))$. As $q \in \pi(F)$, it follows that $Z(K) \leqslant N(\Omega_1(Q))$. So, by (B3) and (B4), $[N_P(\Omega_1(Q)), \alpha] \trianglelefteq N(\Omega_1(Q))$. By (20) and 5.17, as $W \nleqslant F$, $N(\Omega_1(Q)) \leqslant N(P)$.

So $N(Q) \leqslant N(P)$. Thus $Q \in S_q(N(Q))$ and so $Q \in S_q(G)$. As $q \in \sigma_H - \{2\}$, $q \in \pi(F) - \{2, 3\}$. Thus, if $L = O^q(G)$, by 5.21, $q \notin \sigma_L$. But $H \leqslant L$ and so $\sigma_H \subseteq \sigma_L$. This contradiction enables us to conclude that $\sigma_H \subseteq \{2\}$. The lemma is proved.

We now obtain the conclusion of 5.23 when $W \leqslant F$. We must show first that F_3 is cyclic.

LEMMA 5.24. *Suppose $W \leqslant F$. Then F_3 is cyclic.*

Proof. Suppose that $d(F_3) \geqslant 2$. We proceed in a series of steps.

(I) *Let $1 \neq Q \leqslant P$ such that Q is $Z(K)\langle\alpha\rangle$-invariant and $O_{3'}(N(Q)) \leqslant N(P)$. Then $N(Q) \leqslant N(P)$.*

By (A1), $F \leqslant N(P)$. So, by Theorem 6.2.2 of [10], $N_P(Q) \in S_3(N(Q))$. By (B3) and (B4), $N(Q)$ is solvable.

Let $R \in S_3(N(Q))$, $R \geqslant W$. By 4.6(i), and 5.16, $R = N_P(Q)$ and $W \leqslant Z(R)$. By Theorem 6.3.3 of [10],

$$W \leqslant R \cap O_{3',3}(N(Q)) = T \text{ (say)}.$$

Clearly $T \in S_3(O_{3',3}(N(Q)))$. By 4.6(i) and 5.16, $N(T) \leqslant N(P)$. So, by the Frattini argument, as $O_{3'}(N(Q)) \leqslant N(P)$,

$$N(Q) = O_{3',3}(N(Q))(N(Q) \cap N(T)) = O_{3'}(N(Q))(N(Q) \cap N(T)) \leqslant N(P),$$

proving (I).

From (I) and 5.20(v), we have

(II) *$C(a) \leqslant N(P)$ for each $a \in F_3{}^{\#}$. Also $N(F_3) \leqslant N(P)$.*

By 5.10, $O_{3'}(G)$ is solvable. So, by 5.3 and (B4), $2 \in \pi(F)$. Let $S^\alpha = S \in S_2(G)$. By (A1), $F_3 \leqslant N(S)$. So, by (II) and Theorem 6.2.4 of [10], $S \leqslant N(P)$. Let $L = O^2(G)$ and $U = S \cap L$. We note that $[U, F_3] \leqslant P \cap S = 1$, so $F_3 \leqslant C(U)$.

Let $u \in U^{\#}$ and $g \in L$ such that $u^g \in U$. Then $F_3 \leqslant C_L(u^g)$. Let $h = g^{-1}$. Then $F_3{}^h \leqslant C_L(u)$. Also $F_3 \leqslant C_L(u)$. Let $R = C_P(u)$. As $R \geqslant F_3$, by (I) and (II), $N(R) \leqslant N(P)$. So $R \in S_3(N_{C_L(u)}(R))$ and so $R \in S_3(C_L(u))$. We conclude that there is a $c \in C_L(u)$ such that $F_3^{hc} \leqslant R \leqslant P$.

By 4.6(iii), 5.16 and (II), $hc \in N(P)$. So, if $d = c^{-1}$, $dg \in N_L(P)$. As $u^g = u^{dg}$, we have proved

(III) *If $u \in U^{\#}$ and $g \in L$ such that $u^g \in U$, then there is an $n \in N_L(P)$ such that $u^g = u^n$.*

By (III) and Theorem 7.3.4 of [10],

$$U = U \cap L' = U \cap N_L(P)' \leqslant N(P)'.$$

By (B2), $N(P) = PC(P)K$. So, as K is nilpotent, $U \leqslant C_L(P)$. But now, by 5.13 and 5.16, L satisfies the hypotheses of 5.3. Thus $L = O_{3'}(L)\, N_L(P)$.

But, by 5.10, $O_{3'}(L)$ is solvable. So, by (B4), $O_{3'}(L) = 1$ and $P \trianglelefteq L$. Thus $P \trianglelefteq G$, in contradiction with (B4). Thus $d(F_3) = 1$ and F_3 is cyclic. The lemma is proved.

LEMMA 5.25. *Suppose that $W \leqslant F$. Then $W = \Omega_1(F_3) \cong \mathbb{Z}_3$. Let L be an α-invariant proper subgroup of G such that $Z(K) \leqslant L$. Then $R = P \cap L \in S_3(L)$ and L is solvable. Also*

(i) $L = F(L)\, N_L(W)$.

(ii) $L = F(L)\, N_L(P)$.

(iii) *Let $q \in \pi(L)$. Then $[O_q(L), W] \trianglelefteq L$.*

Proof. By 5.16 and (B4), $W \neq 1$. By 5.24, $W = \Omega_1(F_3) \cong \mathbb{Z}_3$. Clearly now $W \leqslant Z(K) \leqslant L$.

Let $R \in S_3(L)$ such that $R \geqslant W$. By 4.6(i) and 5.16, $R = P \cap L$. As $Z(K) \leqslant L$, by (B3), L is solvable. Thus, by Theorem 6.3.3 of [10], if $U = R \cap O_{3',3}(L)$,

$$W \leqslant U \qquad \text{and} \qquad U \in S_3(O_{3',3}(L)).$$

By 4.6(i) and 5.16, $N(U) \leqslant N(W)$. So, by the Frattini argument,

$$L = O_{3',3}(L)\, N_L(U) = O_{3'}(L)\, N_L(U) = O_{3'}(L)\, N_L(W). \tag{21}$$

But, by 2.11, $[O_{3'}(L), W] \leqslant F(L)$. By Theorem 6.2.2 of [10], $O_{3'}(L) = [O_{3'}(L), W]. (O_{3'}(L) \cap C_L(W))$. So, by (21) and 5.16,

$$L = [O_{3'}(L), W]\, N_L(W) = F(L)\, N_L(W) = F(L)\, N_L(P),$$

proving (i) and (ii).

Let $q \in \pi(L)$. As $[O_q(L), W] \trianglelefteq O_q(L)$, $[O_q(L), W] \trianglelefteq F(L)$. But $N_L(W)$ normalizes $[O_q(L), W]$. So, by (i), $[O_q(L), W] \trianglelefteq L$. We have (iii) and the lemma.

LEMMA 5.26. $\sigma_H \subseteq \{2\}$.

Proof. By 5.23, we may suppose $W \leqslant F$. Let $q \in \sigma_H - \{2\}$. As $\sigma_H \subseteq \sigma_G$, $q \in \sigma_G - \{2\}$. By (B2), $N(P) = PC(P)K$ so that $q \in \pi(F)$. Let $Q^\alpha = Q \in S_q(G)$ and $R = N_Q(P)$.

By (A1), $F \leqslant N(Q)$. So, by Theorem 6.2.2 of [10], $R \in S_q(N(P))$. As $F \leqslant N(Q)$, $W \leqslant N(Q)$. So

$$W \leqslant N(Q) \qquad \text{and} \qquad [W, R] \leqslant P \cap Q = 1. \tag{22}$$

Suppose that $C_R(P) \neq 1$. As $K \leqslant N(R)$, $Z(K)$ normalizes $C_R(P)$. So, by (B3), (B4) and (B5), $C(R) \leqslant N(C_R(P)) \leqslant N(P)$. So $C_Q(R) \leqslant Q \cap N(P) = R$. But now, by (22) and Theorem 5.3.4 of [10], $Q \leqslant C(W)$. By 5.16, $Q \leqslant N(P)$. Also $Q = R$ and $C(Q) \leqslant N(P)$. Thus $F(N(Q)) \leqslant N(P)$. But now, by (22) and 5.25(ii), $N(Q) \leqslant N(P)$. Let $L = O^q(G)$.

As $q \in \pi(F) - \{2, 3\}$, by 5.21, $q \notin \sigma_L$. But $H \leqslant L$ and so $\sigma_H \subseteq \sigma_L$, a contradiction. So $C_R(P) = 1$. By (B2), $N(P) = PC(P)K$. By 5.25, $F_3 \neq 1$. So, by (A1), $K = F$. We have proved

$$R = F_q \in S_q(N(P)) \qquad \text{and} \qquad C_R(P) = 1. \tag{23}$$

As $q \in \sigma_H$ there is an $x \in K_q^{\#}$ such that $[C_P(x), \alpha] \neq 1$. Let N be a maximal α-invariant subgroup of G such that $N \geqslant C(x)$. By 5.22, we may suppose $x \in \Omega_1(F_q) \simeq \mathbb{Z}_q$. By Theorems 5.3.4 and 5.3.6 of [10],

$$1 \neq [C_Q(x), \alpha] = [C_Q(x), \alpha, \alpha] \leqslant N. \tag{24}$$

So $N \neq N(P)$. Let $T = P \cap N$. By (A1) and Theorem 6.2.2 of [10], $T \in S_3(N)$. Also, as $x \in F_q$ and F_q is cyclic,

$$F \leqslant C(x) \leqslant N \qquad \text{and} \qquad T \in S_3(N). \tag{25}$$

By (B3), N is solvable. So, by 2.14 and (24), $[C_Q(x), \alpha] \leqslant O_q(N)$. By 2.14, $1 \neq [C_P(x), \alpha] \leqslant O_3(N)$. Suppose that $O_3(N) \cap F_3 \neq 1$. By 5.24 and 5.25, $W = \Omega_1(F_3) \leqslant O_3(N)$. So, by 5.16 and (24), $[C_Q(x), \alpha] \leqslant R$. But now (23) and (24) contradict each other. So $O_3(N) \cap F_3 = 1$. Also $F \leqslant N(T)$. So, by 2.14, $[T, \alpha] \leqslant O_3(N)$. We have proved that

$$1 \neq [T, \alpha] = O_3(N) \qquad \text{and} \qquad O_3(N) \cap F_3 = 1. \tag{26}$$

We now argue that $Q \leqslant N$. Clearly we may suppose that $x \notin Z(Q)$. As $\langle x \rangle = \Omega_1(F_q)$, $C_{Z(Q)}(\alpha) = 1$. But, by (26), 2.3(ii) and Theorem 10.2.1 of [10], $Z(Q) \leqslant C(O_3(N))$.

Let $L = N(Z(Q))$. By 5.25 and (B4), L is solvable. By (A1), $F \leqslant N_L(P \cap L)$. So, by 2.14 and (26), $O_3(N) \leqslant O_3(L)$. By 2.14, $[Q, \alpha] \leqslant O_q(L)$. By (26) and maximality of N, $N = N(O_3(N))$. So, by Theorem 5.3.5 of [10] and (25), $Q = [Q, \alpha] \cdot F_q \leqslant N(O_3(N)) \cdot F = N$. So, in all cases, $Q \leqslant N$.

Let $M = O^q(G)$ and $V = K_\infty(Q \cap M)$. We argue that $F_q \cap M = 1$. Clearly $N(Q) \leqslant N(V)$. Suppose that $V \neq 1$. By 5.25, $N(V)$ is solvable. Now, by 5.16 and (23), $C(W) \cap O_q(N) = F \cap O_q(N)$ and $C(W) \cap O_q(N(V)) = F \cap O_q(N(V))$.

By 2.14, as N and $N(V)$ are solvable, $[Q, \alpha] \leqslant O_q(N) \cap O_q(N(V))$. As $W \leqslant N \cap N(V)$, we find that $1 \neq [Q, \alpha] = [O_q(N), W] = [O_q(N(V)), W]$. By 5.25 and maximality of N, $N(V) \leqslant N([Q, \alpha]) \leqslant N$.

Also, by Glauberman's p-factor theorem [9], as $Q \cap M \in S_q(M)$, $Q \cap M \leqslant O^q(N_M(V)) \leqslant O^q(N)$. But, by 2.14, $[Q, \alpha] = O_q(N)$. So, by 2.6 and Theorem 6.2.2 of [10], $N/O_q(N)$ is q-nilpotent. Thus $Q \cap M \leqslant O_q(N)$. Let $X = (Q \cap M) \cap F$. Then, as $F \leqslant N(P)$,

$$[X, T] \leqslant O_q(N) \cap P = 1.$$

By (26) and maximality of N, $N = N(O_3(N))$. But now $X \leqslant N(P)$ and $C_P(T) \leqslant C_P(O_3(N)) \leqslant P \cap N = T$. So, by Theorem 5.3.4 of [10], $X \leqslant F_q \cap C(P)$. By (23), $X = 1$. So, whether $V = 1$ or $V \neq 1$,

$$F_q \cap M = F_q \cap (Q \cap M) = 1.$$

But now $q \notin \sigma_M$. As $H \leqslant M$, $\sigma_H \subseteq \sigma_M$, a contradiction. Thus no such q can exist and $\sigma_H \subseteq \{2\}$. The lemma is proved.

We next show that, when $2 \in \sigma_H$, P is not abelian. This means that $C_P(t) \neq 1$ for each involution t of $N(P)$. This in turn enables us to show that $N(P)$ contains a Sylow 2-subgroup of G. In the case $W \nleqslant F$ we get a contradiction using a transfer argument. In the case $W \leqslant F$, an argument resembling the proof of 5.26 enables us to conclude that $\sigma_H = \emptyset$.

LEMMA 5.27. *Suppose that $2 \in \sigma_H$. Then P is not abelian.*

Proof. Suppose that P is abelian. Then, by (B5) and Theorem 5.2.3 of [10], $F_3 = 1$. Set $V = \Omega_1(P)$. Clearly V is weakly closed in P with respect to G. By (B3) and (B4), $N(V) = N(P)$. As $P = Z(J(P))$, $N(P) = N(Z(J(P)))$. By (B2), $N(P) = PC(P)K$. Also, by Theorem 5.2.4 of [10], $C(V) = C(P)$.

Let $q \in \pi(G) - \{3\}$. Using the notation of 5.4, we have from 5.4(2) that there is a $Q \in \mathcal{U}_G^*(V, q)$ such that $Q = Q^\alpha$. We claim that $Q \leqslant C(V)$.

As $2 \in \sigma_H \subseteq \sigma_G$, by 5.19, $K = F$. So $C_{QV}(\alpha) \leqslant N(V)$. As $F_3 = 1$, by 2.14, $V = [V, \alpha] \leqslant O_3(QV)$ and so $Q \leqslant C(V)$. By 5.4(1), V centralizes every V-invariant q-subgroup of G for each $q \in \pi(G) - \{3\}$. By Theorem 6.2.2 of [10], each V-invariant $3'$-subgroup of G is centralized by V.

As $F_3 = 1$, by 5.17, 5.16 and what we have proved above, V, G and P satisfy the hypotheses of 4.8. Using the notation of 4.8, we have that G, P, π, U and W_X satisfy Hypothesis 4.1, $W_1 = V$ and $N(P) = PC(P)U$.

By 5.10, $O_{3'}(G)$ is solvable. So, by (B4) and (B5), $G \neq O_{3'}(G)\,N(P)$. Thus, by 5.3, $2 \in \pi$.

Let $T^\alpha = T \in S_2(N(P))$. Suppose that $C_T(P) \neq 1$. As $2 \in \pi(F)$, by (A1), (B3) and (B4), $N(C_T(P)) \leqslant N(P)$. So, by 4.2, $P \trianglelefteq G$, in contradiction with (B4) and (B5).

We conclude that $C_T(P) = 1$. By (B2), $N(P) = PC(P)K$. So $T = K_2$. But, as $2 \in \sigma_H \subseteq \sigma_G$, there is an $x \in K_2^{\#}$ such that $C_P(x) \neq 1$. By 5.19, $K = F$. So, by Theorem 6.2.2 of [10], $C_P(x) \in S_3(C(x))$. Also $Z(K) \leqslant C(x)$. So, by (B3) and (B4), as $F_3 = 1$, $C_P(x) \trianglelefteq C(x)$. By 5.17, $C(x) \leqslant N(P)$.

Let $S^\alpha = S \in S_2(G)$ such that $S \geqslant K_2$. Then $S \cap N(P) = K_2$. So, we have that, $C_S(K_2) \leqslant C_S(x) \leqslant S \cap N(P) = K_2$.

So, by Theorem 5.3.4 of [10], $S \leqslant F$. But now, by 2.6 and the Feit-Thompson theorem [5], G is solvable. By (B4) and (B5), we have a contradiction. The lemma now follows.

LEMMA 5.28. *Suppose that* $F_3 \neq 1$. *Then* $K_2 \in S_2(N(P))$ *and* $K_2 \cap C(P) = 1$.

Proof. Let $T^\alpha = T \in S_2(N(P))$ and $S^\alpha = S \in S_2(G)$ such that $S \geqslant T$. By 5.10, $O_{3'}(G)$ is solvable. So, by (B4) and (B5), $G \neq O_{3'}(G)\,N(P)$. By 5.16 and 5.3, $K_2 \neq 1$ and so $2 \in \pi(F)$. So, by (A1), $Z(K)$ normalizes T.

Suppose that $C_T(P) \neq 1$. Then $Z(K) \leqslant N(C_T(P))$. By (B3) and (B4), $N(C_T(P)) \leqslant N(P)$. So $C(T) \leqslant N(P)$. Now, by (A1), F_3 normalizes T. So $(F_3, T] \leqslant P \cap T = 1$ and $F_3 \leqslant C_P(T)$.

By 5.16 and 4.6(i), if $W \leqslant F$, then $N(C_P(T)) \leqslant N(P)$. If $W \nleqslant F$ then, by 5.17, $N(C_P(T)) \leqslant N(P)$. So, in all cases, $N(C_P(T)) \leqslant N(P)$. Thus $C_P(T) \in S_3(N_{C(T)}(C_P(T)))$ and so $C_P(T) \in S_3(C(T))$. By the Frattini argument, $N(T) = C(T)(N(C_P(T)) \cap N(T)) \leqslant N(P)$.

So $T \in S_2(N(T))$ and so $T \in S_2(G)$. So $T = S$. Let $L = O^2(G)$ and $U = T \cap L$. Then $U \in S_2(L)$ and $U \leqslant N_L(P)$.

Let $t \in U^{\#}$ and $g \in L$ such that $t^g \in U$. Then, if $h = g^{-1}$, $F_3{}^h \leqslant C(t)$. By (B2), $N(P) = PC(P)K$. So, by Theorem 5.3.5 of [10], $U = [U, \alpha] \cdot C_U(\alpha) = C_U(P)\,C_U(\alpha)$. Thus $t = vw$ where $v \in C_U(P)$ and $w \in C_U(\alpha)$.

We conclude that $C_P(t) = C_P(w)$, a $Z(K)\langle\alpha\rangle$-invariant subgroup of P. By 5.17, if $W \nleqslant F$, $N(C_P(t)) \leqslant N(P)$. So, by 5.16 and 4.6(i) (for the case $W \leqslant F$), $N(C_P(t)) \leqslant N(P)$.

Thus $C_P(t) \in S_3(N(C_P(t)) \cap C_L(t))$. So $C_P(t) \in S_3(C_L(t))$. There is therefore a $c \in C_L(t)$ such that $F_3^{hc} \leqslant C_P(t)$. Set $d = c^{-1}$. Then, by 5.16 and 4.6(iii), there is an $n \in N(P)$ such that $ndg \in N(F_3)$. As before, $N(F_3) \leqslant N(P)$. So $dg \in N(P)$. As $t^g = t^{dg}$, we have proved

(*) *If* $g \in L$ *and* t, $t^g \in U$ *then there is an* $n \in N_L(P)$ *such that* $t^g = t^n$.

By Theorem 7.3.4 of [10], $U = U \cap L' = U \cap N_L(P)' \leqslant N(P)'$. By (B2), $N(P) = PC(P)K$. So $U \leqslant C_L(P)$. By 5.10, $O_{3'}(L) \leqslant O_{3'}(G)$, a solvable group. So, by (B4), $O_{3'}(L) = 1$. Thus, by (B4) and (B5), $L \neq O_{3'}(L)\, N_L(P)$. By 5.13 and 5.16, 5.3 applies to L. So $U \nleqslant C_L(P)$, a contradiction.

We deduce that $C_T(P) = 1$. As $N(P) = PC(P)K$, $T = K_2 \in S_2(N(P))$ and $C_T(P) = K_2 \cap C(P) = 1$. The lemma is proved.

LEMMA 5.29. *Suppose that* $2 \in \sigma_H$. *Then* $W \leqslant F$.

Proof. Suppose that $W \nleqslant F$. We prove this lemma in a series of steps, ending in a transfer argument. Let $T^\alpha = T \in S_2(N(P))$.

(I) *Let t be an involution of T. Then* $N(C_P(t)) \leqslant N(P)$ *and* $C_P(t) \in S_3(C(t))$.

By 5.27, P is not abelian. So $C_P(t) \neq 1$. By (B2), $N(P) = PC(P)K$. So $T = C_T(P)K_2$. We deduce that $t = uv$ where $u \in C_T(P)$ and $v \in K_2$. So $C_P(t) = C_P(v)$. We conclude that $1 \neq C_P(t) = C_P(v)$ is $Z(K) \cdot \langle\alpha\rangle$-invariant. By 5.17, $N(C_P(t)) \leqslant N(P)$. So $C_P(t) \in S_3(N_{C(t)}(C_P(t)))$ and $C_P(t) \in S_3(C(t))$. We have (I).

(II) *Let* $t \in F_2{}^\#$. *Then* $C(t) \leqslant N(P)$. *Also* $F_3 = 1$ *and* $K = F$.

By 5.19, as $\sigma_H \neq \emptyset$, $K = F$. Let $x \in K_2{}^\#$ such that $[C_P(x), \alpha] \neq 1$. We may suppose $o(x) = 2$. By (I), as $Z(K) \leqslant C(x)$, and by (B3) and (B4), $[C_P(x), \alpha] \trianglelefteq C(x)$. By 5.17, as $W \nleqslant F$, $C(x) \leqslant N(P)$.

If $T = F_2$ and $U^\alpha = U \in S_2(G)$, $U \geqslant T$, then $C_U(F_2) \leqslant C_U(x) \leqslant F_2$. So, by Theorem 5.3.4 of [10], $U = F_2$. So, by 2.6, G is 2-nilpotent. By the Feit-Thompson theorem [5], G is solvable. But, by (B4), we have a contradiction. So $F_2 \notin S_2(N(P))$. By 5.28, $F_3 = 1$.

Let $t \in F_2{}^\#$. Then, in order to show that $C(t) \leqslant N(P)$, we may suppose $o(t) = 2$. As $K = F$, t is an involution of T. By (I), $C_P(t) \in S_3(C(t))$. Also $Z(K) \leqslant C(t)$. By (B3) and (B4), as $F_3 = 1$, $C_P(t) \trianglelefteq C(t)$. So, by (I), $C(t) \leqslant N(P)$. We have (II).

(III) *Let* $t \in F_2{}^\#$ *and* $g \in G$ *such that* $t^g \in T$. *Then* $g \in N(P)$.

We may suppose $o(t) = 2$. Let $h = g^{-1}$. By (II), $C_P(t^g)^h \leqslant C(t) \leqslant N(P)$. So $C_P(t^g)^h \leqslant P$. By 5.16 and 4.6(iii), there is an $n \in N(P)$ such that $hn^{-1} \in N(C_P(t^g))$. So, by (I), $h \in N(P)$ and so $g \in N(P)$. We have (III).

(IV) $F_2 \neq 1$ *and* $T \in S_2(G)$.

As $2 \in \sigma_H \subseteq \sigma_G$, $2 \in \pi(F)$ and $F_2 \neq 1$. So, by (III), $N(T) \leqslant N(P)$. But now $T \in S_2(N(T))$ and so $T \in S_2(G)$. We have (IV).

Let $L = O^2(G)$ and $S = T \cap L$. Then $S \in S_2(L)$. By 5.13 and (B2), $N_L(P) = PC_L(P)(K \cap L)$. So, by Theorem 5.3.5 of [10],

$$S = C_S(P) \cdot (K_2 \cap S). \tag{27}$$

As $H \leqslant L$, $\sigma_H \subsetneq \sigma_L$. So $2 \in \sigma_L$ and $K_2 \cap S \nleqslant C(P)$. Let $V = S'C_S(P)$. As $S' \leqslant \Phi(S)$, $V \neq S$. Let θ denote the transfer homomorphism $L \to S/V$ and ϕ denote the natural map $S \to S/V$. Then, by Theorem 7.3.3 of [10], if $t \in K_2 \cap S$,

$$t\theta = \prod_{i=1}^{n} (x_i t^{r_1} x_i^{-1})\phi,$$

for r_i, $n \in \mathbb{Z}$ and $x_i \in L$. Also $x_i t^{r_i} x_i^{-1} \in S$, $1 \leqslant i \leqslant n$. Suppose that $t^{r_i} = 1$. Then $x_i t^{r_i} x_i^{-1} = (t\phi)^{r_i}$. Suppose next that $t^{r_i} \neq 1$. By (III), $x_i \in N_L(P)$. As $N_L(P) = PC_L(P)(K \cap L)$,

$$t^{-r_1} x_i t^{r_1} x_i^{-1} \in (PC_L(P)\, S') \cap S = V.$$

So, in all cases, $(x_i t^{r_i} x_i^{-1})\phi = (t\phi)^{r_i}$. So $t\theta = (t\theta)^{\Sigma r_i} = (t\phi)^{|L:S|}$. But $L/\mathrm{Ker}\,\theta$ is a 2-group. As $L = O^2(L)$, $L = \mathrm{Ker}\,\theta$. As $|L : S|$ is odd, $K_2 \cap S \leqslant V$. So, by (27), $S = V$, a contradiction. So $W \leqslant F$ and the lemma is proved.

Lemma 5.30. $\sigma_H = \emptyset$.

Proof. We proceed in a series of steps. Suppose that $\sigma_H \neq \emptyset$. By 5.26, $\sigma_H = \{2\}$. By 5.29, $W \leqslant F$. So 5.25 applies. By 5.25, $W \neq 1$. So $F_3 \neq 1$. By (A1), $K = F$. By 5.28, $K_2 \cap C(P) = 1$ and $K_2 \in S_2(N(P))$. As $\sigma_H = \{2\}$ there is an involution $t \in K_2$ such that $[C_P(t), \alpha] \neq 1$.

Let L be a maximal α-invariant subgroup of G such that $L \geqslant C(t)$. Let $T^\alpha = T \in S_2(G)$, $T \geqslant K_2$. Suppose that $C(t) \leqslant N(P)$. Then $C_T(K_2) \leqslant N(P) \cap T = K_2$. So, by Theorem 5.3.4 of [10], $T = K_2$. Thus, by 2.6, G is 2-nilpotent. By the Feit-Thompson theorem [5], G is solvable, in contradiction with (B4). So we have proved

$$C(t) \nleqslant N(P) \qquad \text{and} \qquad L \neq N(P). \tag{28}$$

As $K = F$, by Theorem 6.2.2 of [10], $C_P(t) \in S_3(C(t))$. By 5.25, L is solvable and $P \cap L \in S_3(L)$. So, by 2.14,

$$1 \neq [C_P(t), \alpha] \leqslant [P \cap L, \alpha] \leqslant O_3(L). \tag{29}$$

Suppose that $O_3(L) \cap F \neq 1$. By 5.25, $W \leqslant O_3(L)$. So, by 4.6(i) and 5.16, $W \trianglelefteq L$ and $L \leqslant N(P)$, in contradiction with (28). So

$$O_3(L) \cap F = 1. \tag{30}$$

We next show that $T \leqslant L$. In order to do this we must prove

(I) *F_2 is cyclic or generalized quaternion.*

Recall that $K = F$. Suppose that $d(F_2) \geqslant 2$. Then, by 4.7 and (B5),

$$P = \langle [C_P(x), \alpha]: x \in F_2^{\#} \rangle.$$

Let $x \in F_2^{\#}$. Then $Z(K) \leqslant C(x)$. By Theorem 6.2.2 of [10], as $K = F$, $C_P(x) \in S_3(C(x))$. By (B3) and (B4), $C(x)$ is solvable. So, by 2.14, $[C_P(x), \alpha] \leqslant O_3(C(x))$. So

$$P = \langle O_3(C(x)): x \in F_2^{\#} \rangle. \tag{31}$$

Let $x \in F_2^{\#}$. By 5.25, $W \leqslant Z(F) \leqslant C(x)$. Also, by (B4), G is not solvable. So, by 2.6 and the Feit-Thompson theorem [5], $T \nleqslant F$. By Theorem 5.3.4 of [10], $Z = [C_T(F_2), \alpha] \neq 1$. So $Z \leqslant C(x)$. As $K_2 \in S_2(N(P))$, and $W \leqslant C(x)$, by 5.25(ii), $Z = [Z, W]$. So, by 2.13, $Z \leqslant O_2(C(x))$. So $[O_3(C(x)), Z] = 1$. We deduce from (31) that $[P, Z] = 1$. But $K_2 = T \cap N(P)$. So $Z \leqslant T \cap N(P) = K_2$. By Theorem 5.3.6 of [10], we have a contradiction. Thus $d(F_2) = 1$ and we have (I).

(II) $T \leqslant L$. *Also* $F \leqslant L$.

As $o(t) = 2$, by (I), $\langle t \rangle = \Omega_1(F_2)$. As $L \geqslant C(t)$, we may suppose that $C_{Z(T)}(\alpha) = 1$. Also $F \leqslant C(t) \leqslant L$. Now $Z(T) \leqslant C(t)$.

By (30) and 2.3(ii), α acts fixed point freely on $O_3(L) \cdot Z(T)$. By Theorem 10.2.1 of [10], $O_3(L) \leqslant C(Z(T))$. It follows from 5.25(ii), as $K_2 \in S_2(N(P))$, that $[T, \alpha] \leqslant O_2(N(Z(T)))$. By 5.25, $N_P(Z(T)) \in S_3(N(Z(T)))$ and $N(Z(T))$ is solvable.

By 2.14 and (30), $O_3(L) \leqslant [N_P(Z(T)), \alpha] \leqslant O_3(N(Z(T)))$. We deduce that $[T, \alpha] \leqslant C(O_3(L))$. But, by (29) and maximality of L, $[T, \alpha] \leqslant N(O_3(L)) = L$. As $F \leqslant L$, by Theorem 5.3.5 of [10], $T = [T, \alpha]F_2 \leqslant L$. We have (II).

By 5.25(ii), as $W \leqslant L$ and $K_2 \in S_2(N(P))$, $[T, \alpha] \leqslant O_2(L)$. Let $X = O_2(L) \cap F$. As $F = K$, $X \leqslant N(P)$. By 5.25, $P \cap L \in S_3(L)$. But $[P \cap L, X] \leqslant P \cap O_2(L) = 1$.

By (29) and maximality of L, $L = N(O_3(L))$. So

$$C_P(C_P(X)) \leqslant C_P(P \cap L) \leqslant C_P(O_3(L)) \leqslant P \cap L \leqslant C_P(X).$$

By Theorem 5.3.4 of [10], $P = C_P(X)$. But now, by 5.28, $X = 1$. So we have proved that

$$[T, \alpha] = O_2(L) \qquad \text{and} \qquad O_2(L) \cap F = 1. \tag{32}$$

We now prove

(III) *Let S be a non-identity $Z(K) \cdot \langle \alpha \rangle$-invariant subgroup of $O_2(L)$. Then $N(S) \leqslant L$. Also $[T, \alpha] \neq 1 \neq F_2$ and $F \leqslant N(T)$.*

As in the proof that $Z \neq 1$ in (I), $[T, \alpha] \neq 1$. Let S be maximal subject to violating (III). By (B3) and (B4), $N(S)$ is solvable. As $2 \in \sigma_H$, $2 \in \pi(F)$ and so $F_2 \neq 1$. By (A1), $F \leqslant N(T)$. As $W \leqslant Z(K)$, W normalises S. So $W \leqslant N_G(N_T(S))$.

By Theorem 6.2.2 of [10], $N_T(S) \in S_2(N(S))$. By 5.16, $K_2 \in S_2(N(W))$. So, by 2.13, $[N_T(S), W] \leqslant [O_2(N(S)), W] \leqslant [N_T(S), W]$ and $[N_T(S), W] = [N_T(S), W] = [N_T(S), \alpha]$. By (32), $[N_T(S), \alpha] \gneqq S$. So, by maximality of S, and 5.25(iii), $N(X) \leqslant N([N_T(S), W]) = N([N_T(S), \alpha]) \leqslant L$, a contradiction. We have (III).

(IV) *Let M be an α-invariant proper subgroup of G such that $Z(K) \leqslant M$ and $[T \cap M, \alpha] \neq 1$. Then $M \leqslant L$.*

As $Z(K) \leqslant M \lneqq G$, by 5.25, M is solvable. By (III), $F \leqslant N(T)$. So, by Theorem 6.2.2 of [10], $T \cap M \in S_2(M)$. By 5.16, $K_2 \in S_2(N(W))$. So $1 \neq [T \cap M, \alpha] = [T \cap M, W]$. Also, by 2.13, $[T \cap M, W] \leqslant [O_2(M), W] \leqslant [T \cap M, W]$. So, by 5.25(iii), $1 \neq [O_2(M), W] \trianglelefteq M$. Also, by (32), $[O_2(M), W] \leqslant O_2(L)$. So, by (III), $M \leqslant L$. We have (IV).

Let $Y = \Omega_1(Z(T)) \cap O_2(L)$. We argue that Y is weakly closed in T with respect to G. By (III) and (32), $Y \neq 1$. As, by (II) and (III), $F \leqslant N(T) \cap L$, we have that $F \leqslant N(Y)$. By (III), $N(Y) \leqslant L$.

Let $q \in \pi(L) - \{2\}$ and $Q^\alpha = Q \in S_q(L)$. If $q \notin \pi(F)$ then, by (32) and 2.3(ii), α acts fixed point freely on $O_2(L) \cdot Q$. So, by Theorem 10.2.1 of [10], $Q \leqslant C(O_2(L))$. But, if $q \in \pi(F)$, then, by Theorem 6.2.2 of [10] and (A1), $F \leqslant N_L(Q)$. So, by 2.14, as L is solvable, $[Q, \alpha] \leqslant O_q(L) \leqslant C(O_2(L))$.

We have therefore proved that if $q \in \pi(L) - \{2\}$ and $Q^\alpha = Q \in S_q(L)$ then $[Q, \alpha] \leqslant C_L(O_2(L)) \trianglelefteq L$. So, as $F \leqslant L$, $L = C_L(O_2(L)) \cdot T \cdot F$. By maximality of L, $L = N(O_2(L))$. By definition of Y, $Y \trianglelefteq L$. So, as $N(Y) \leqslant L$, we have proved that

$$L = N(O_2(L)) = C(O_2(L)) \cdot T \cdot F \qquad \text{and} \qquad N(Y) = L. \tag{33}$$

By Theorem 5.3.5 of [10], $T = [T, \alpha] \cdot F_2$. By (32), $T = O_2(L) \cdot F_2$ and $O_2(L) \cap F_2 = 1$. By (I), $\Omega_1(T) \leqslant O_2(L) \cdot \langle t \rangle$. So, by (29), as $C_P(t)$ is α-invariant,

$$U = O_3(L) \cap C(\Omega_1(T)) \neq 1. \tag{34}$$

Now, as $F \leqslant L \cap N(T)$, $F \leqslant N(U)$. Also, $O_2(L) \leqslant C(U)$. By (32) and (III), $[N_T(U), \alpha] \neq 1$. So, by (34), (IV) and (B4),

$$N(U) \leqslant L. \tag{35}$$

Let $x \in O_2(L)^{\#}$. We argue that $O_3(L) \in S_3(C_L(x))$. Let $R \in S_3(C_L(x))$, $R \geqslant O_3(L)$. Suppose that $R \neq O_3(L)$. By 5.25, $P \cap L \in S_3(L)$. So there is an $l \in L$ such that $O_3(L) \leqslant R^l \leqslant P \cap L$.

By Theorem 5.3.5 of [10], $P \cap L = O_3(L) \cdot F_3$. As $R \neq O_3(L)$, $R^l \cap F_3 \neq 1$. So, by 5.25, $W \leqslant R^l \leqslant C(x^l)$. So $O_2(L) \cap C(W) \neq 1$. But, as $K_2 \in S_2(N(P))$, by 5.16, $K_2 \in S_2(N(W))$. So $1 \neq O_2(L) \cap C(W) \leqslant O_2(L) \cap F_2$. This contradicts (32). We have therefore proved

(V) *Let $x \in O_2(L)^{\#}$. Then $O_3(L) \in S_3(C_L(x))$.*

We also require

(VI) $T = O_2(L) \cdot F_2$. *Furthermore if $g \in G$ such that $O_2(L)^g \leqslant L$ then $g \in L$.*

As remarked after (33), $T = O_2(L) \cdot F_2$. Let $g \in G$ such that $O_2(L)^g \leqslant L$ and $h = g^{-1}$. Then $O_2(L) \leqslant T^h \cap L = T_0$ (say). Now there is an $l \in L$ such that $O_2(L) \leqslant T_0{}^l \leqslant T$. As $T = O_2(L) \cdot F_2$, $T_0{}^l = O_2(L) \cdot (F_2 \cap T_0{}^l)$. So $T_0{}^l$ is α-invariant and $Z(F) \leqslant N(T_0{}^l)$. As $K = F$, $Z(K) = Z(F) \leqslant N(T_0{}^l)$. So, by (32) (III) and (IV), $N(T_0{}^l) \leqslant L$. Thus $N(T_0) \leqslant L$. But now

$$N_{T^h}(T^h \cap L) \leqslant N_{T^h}(T_0) \leqslant T^h \cap L.$$

So $T^h \leqslant L$. Thus there is an $m \in L$ such that $T^h = T^m$. So $mg \in N(T)$. But, by (III) and (IV), $N(T) \leqslant L$. So, $g \in L$. We have (VI).

We can now show that Y is weakly closed in T with respect to G. Let $g \in G$ such that $Y^g \leqslant T$ and set $h = g^{-1}$. As Y is elementary abelian, $Y^g \leqslant \Omega_1(T)$. So $U \leqslant C(Y^g)$. By (33), $U^h \leqslant C(Y) \leqslant L$. So $U^h \leqslant C_L(Y)$. By (V), as $Y \leqslant O_2(L)$, $U^h \leqslant O_3(L)$.

So $O_2(L) \leqslant C(U^h) = C(U)^h$. Thus, by (35), $O_2(L)^g \leqslant C(U) \leqslant L$. By (VI), $g \in L$. But finally, by (33), $Y^g = Y$. So Y is weakly closed in T with respect to G.

By Theorem 14.4.2 of [15] and (33),

$$T \cap O^2(G) = T \cap O^2(L).$$

But, by 2.6 and (32), $L/O_2(L)$ is 2-nilpotent. So $T \cap O^2(L) \leqslant O_2(L)$. We conclude that $O_2(L) \cap O^2(G) \in S_2(O_2(G))$. But now, by (32), $2 \notin \pi(F \cap O^2(G))$. However, $2 \in \sigma_H$ and $H \leqslant O^2(G)$. This is a contradiction. So $\sigma_H = \emptyset$ and the lemma is proved.

We now show that $C_H(a) \leqslant N_H(P)$ for each $a \in P^{\#}$. We then apply 4.3 to get a contradiction. For $a \in G$, set $a^{N(P)} = \{a^n : n \in N(P)\}$.

LEMMA 5.31. *Let $a \in P^{\#}$. Then $C_H(a) \leqslant N_H(P)$. Also, if $a^{N(P)} \cap F_3 = \emptyset$ then $C_H(a) \leqslant PC_H(P)$.*

Proof. We proceed in a series of steps. Let $a \in P^{\#}$. We consider two cases as $a^{N(P)} \cap F_3 \neq \emptyset$ and $a^{N(P)} \cap F_3 = \emptyset$.

(I) *Suppose that $a^{N(P)} \cap F_3 \neq \emptyset$. Then $C(a) \leqslant N(P)$.*

Let $n \in N(P)$ such that $a^n \in F_3$. If $W \nleqslant F$ then, by 5.17, $C(a^n) \leqslant N(P)$. If $W \leqslant F$ then, by 5.25, $W \leqslant \langle a^n \rangle$ and so, by 5.16, $C(a^n) \leqslant N(P)$. As $C(a^n) = C(a)^n$, we have (I).

In the next step we use 5.30.

(II) *Suppose that* $a^{N(P)} \cap F_3 = \emptyset$. *Then* $C_H(a) \cap N_H(P) \leqslant PC_H(P)$ *and* $C_H(a) = O_{3'}(C_H(a)) \cdot C_P(a)$.

Set $C = C_H(a) \cap N_H(P)$. By (B2) and 5.13, as $P \leqslant H$, $N_H(P) = PC_H(P)(K \cap H)$. Suppose that $C \nleqslant PC_H(P)$. Let $q \in \pi(CP/PC_H(P))$. We argue that $q \in \sigma_H$.

Certainly $q \in \pi(\mathrm{Aut}_H(P)) - \{3\}$. Let $Q \in S_q(C)$. Then $C_Q(P) \in S_q(C_H(P))$. Let $R^\alpha = R \in S_q(N_H(P))$. There is an $n \in N(P)$ such that $Q^n \leqslant R$. Also $Q^n \cap C(P) = R \cap C(P) \in S_q(C_H(P))$.

But, as $N_H(P) = PC_H(P)(K \cap H)$, by Theorem 5.3.5 of [10], $R = C_R(P) \cdot K_r$. As $Q \nleqslant C_H(P)$, $Q^n \cap K_r \nleqslant C_H(P)$. Let $x \in (Q^n \cap K_r) - C_H(P)$. Then $a^n \in C_P(x)$. As $a^{N(P)} \cap F_3 = \emptyset$, $[C_P(x), \alpha] \neq 1$. But now $q \in \sigma_H$, which contradicts 5.30. So

$$C \leqslant PC_H(P) = P \times O_{3'}(N_H(P)), \text{ a 3-nilpotent group.} \tag{36}$$

Let $D = C_H(a)$. Now $W \leqslant C_P(a)$. So, by 4.6(i) and 5.16, $N(C_P(a)) \leqslant N(P)$. So $C_P(a) \in S_3(N_D(C_P(a)))$ and so $C_P(a) \in S_3(D)$. By 4.6(i), $W \leqslant Z(C_P(a)) \leqslant Z(J(C_P(a)))$. So, by 5.16, 4.6(i) and (36),

$$N_D(Z(J(C_P(a)))) \leqslant D \cap N_H(P) = C, \text{ a 3-nilpotent group.}$$

But now, by Theorem 8.3.1 of [10], $D = O_{3'}(D) \cdot C_P(a)$. We have proved (II).

Using (I) and (II), we have that

(III) *Let* $a \in P^\#$. *Then* $C_H(a) = O_{3'}(C_H(a)) \cdot (N_H(P) \cap C_H(a))$.

(IV) *Let* $A \in SCN_3(P)$ *and* $b \in P^\#$ *such that* $m(C_A(b)) \geqslant 2$. *Set* $X_A = \langle O_{3'}(C_H(a)) : a \in A^\# \rangle$. *Then* X_A *is a* 3'*-subgroup of* G *and* $O_{3'}(C_H(b)) \leqslant X_A$.

By (B3), (B4) and (B5), $N(P)$ is solvable. So, by (III), $C(a)$ is 3-solvable for each $a \in P^\#$. Using the Proposition on p. 90 of [22], we see that $O_{3'}(C_H(a))$ is an A-signalizer functor on G. By a theorem of McBride [19], as $m(A) \geqslant 3$, X_A is a 3'-subgroup of G.

Let $b \in P^\#$ such that $m(C_A(b)) \geqslant 2$. Then, by Theorem 6.2.4 of [10], $O_{3'}(C_H(b)) = \langle O_{3'}(C_H(b)) \cap C_H(a) : a \in C_A(b)^\# \rangle$. But, by the argument on p. 90 of [22], $O_{3'}(C_H(b)) \leqslant \langle O_{3'}(C_H(a)) : a \in C_A(b^\#) \rangle \leqslant X_A$.

(V) *Let* $A \in SCN_3(P)$. *Then we may suppose that there is a* $b \in P^\#$ *such that* $m(C_A(b)) = 1$ *and* $o(b) = 3$.

Let X_A be in (IV). Suppose that $m(C_A(b)) \geqslant 2$ for each $b \in P^{\#}$ such that $o(b) = 3$.

Let $b \in P^{\#}$ and $c \in \Omega_1(\langle b \rangle)^{\#}$. By 4.5 and (III), as $C_H(c)$ is 3-solvable, $O_{3'}(C_H(b)) \leqslant O_{3'}(C_H(c))$. But $m(C_A(c)) \geqslant 2$. So, by (IV), $O_{3'}(C_H(b)) \leqslant O_{3'}(C_H(c)) \leqslant X_A$. We deduce that

$$X_A = \langle O_{3'}(C_H(b)) \colon b \in P^{\#} \rangle.$$

So X_A is α-invariant and $N(P) \leqslant N(X_A)$. By 5.10 and (IV), X_A is solvable. By (III), we may suppose $X_A \nleqslant N(P)$. But, by (B3), (B4) and (B5), $X_A \leqslant N(X_A) \leqslant N(P)$, a contradiction. We have (V).

(VI) *$SCN_4(P) = \emptyset$, P has nilpotency class greater than* 2 *and* $W \leqslant F$.

Let $A \in SCN_4(P)$ and b be as in (V). By considering the Jordan canonical from of b acting on $\Omega_1(A)$, we see that $m(C_A(b)) > 1$, a contradiction. So $SCN_4(P) = \emptyset$.

Suppose that P has nilpotency class at most 2. Let $A \in SCN_3(P)$ and b be as in (V). Then the Jordan canonical form of b on $\Omega_1(A)$ cannot contain a block like

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

as $[\Omega_1(A), b, b] = 1$. But this means that $m(C_A(b)) \geqslant 2$, a contradiction.

As $p \geqslant 5$, by (A3), if $W \nleqslant F$ then $m(W) \geqslant 3$. But $W \leqslant A \in SCN(P)$. As $W \leqslant Z(P)$, this contradicts (V).

(VII) *Every abelian characteristic subgroup of P is cyclic.*

Let A be an abelian characteristic subgroup of P. By 5.24, we may suppose $A \nleqslant F_3$. But, by Theorem 5.2.3 of [10], as $p \geqslant 5$, $m([A, \alpha]) \geqslant 3$. But $AW \leqslant B \in SCN(P)$. Clearly $B \geqslant [A, \alpha] \times W$. But now, $m(B) \geqslant 4$, in contradiction with (VI). We have (VII).

By (VII) and Theorem 5.4.9 of [10], P has nilpotency class 2. This contradicts (VI). So $C_H(a) \leqslant N_H(P)$ for each $a \in P^{\#}$. By (II), we see that if $a^{N(P)} \cap F_3 = \emptyset$ $C_H(a) \leqslant PC_H(P)$. The lemma is proved.

LEMMA 5.32. *Let $\pi = \pi(N_H(P)/PC_H(P))$. Then*

(i) *$N(P)$ is solvable.*

(ii) *H is not solvable.*

(iii) $H \neq O_{3'}(H) \cdot N_H(P)$.

(iv) $2 \in \pi$.

(v) *Let Q be a non-identity α-invariant subgroup of P such that $Q \geqslant \Omega_1(Z(P)) \cap F$. Then $N(Q) \leqslant N(P)$.*

(vi) *$O_\pi(K \cap H)$ is a Hall π-subgroup of $N_H(P)$.*

Proof. (i) follows from (B3), (B4) and (B5).

By (B5), $1 \neq P \leqslant H \trianglelefteq G$. By (B4), $F(G) = 1$ and so $F(H) = 1$. But now H cannot be solvable. We have (ii).

By 5.10, $O_{3'}(H)$ is solvable. So, by (i), $O_{3'}(H) \cdot N_H(P)$ is solvable. By (ii), we have (iii).

By (B2) and 5.13, $N_H(P) = PC_H(P)(K \cap H)$. By 5.16, H, P, W satisfy Hypothesis 5.2. But now, if $2 \notin \pi$, by 5.3, $H = O_{3'}(H) \cdot N_H(P)$. By (iii), $2 \in \pi$ and we have (iv).

Let $1 \neq Q$ be a $Z(K) \cdot \langle\alpha\rangle$-invariant subgroup of P such that $Q \geqslant \Omega_1(Z(P)) \cap F$. By 5.17, if $W \nleqslant F$ then $N(Q) \leqslant N(P)$. If $W \leqslant F$ then, by 5.25, $W \leqslant Q$. But now, by 4.6(i) and 5.16, $N(Q) \leqslant N(P)$. We have (v).

Suppose that $F_3 \neq 1$. Let $q \in \pi$ and $R^\alpha = R \in S_q(N(P))$. Suppose that $C_R(P) \neq 1$. By (B3), (B4) and (B5), $N(C_R(P)) \leqslant N(P)$. So

$$C(R) \leqslant N(C_R(P)) \leqslant N(P). \tag{37}$$

Clearly $q \in \pi(F)$. By (A1) and Theorem 6.2.2 of [10], $K \leqslant N(R)$. So $[F_3, R] \leqslant R \cap P = 1$ and $C_P(R) \neq 1$. By (v), $N(C_P(R)) \leqslant N(P)$. So, by the Frattini argument and (37),

$$N(R) = C(R) \cdot (N(C_P(R)) \cap N(R)) \leqslant N(P).$$

So $R \in S_q(N(R))$ and $R \in S_q(G)$. By 5.28, as $C_R(P) \neq 1$ and $F_3 \neq 1$, $q > 3$. Let $L = O^q(G)$. By 5.21, $q \notin \pi(\mathrm{Aut}_L(P))$. But $H \leqslant L$ and so $\pi(\mathrm{Aut}_H(P)) \subseteq \pi(\mathrm{Aut}_L(P))$, a contradiction. So $C_R(P) = 1$. But, by (B2) and 5.13, $N_H(P) = PC_H(P)(K \cap H)$. So $R \leqslant K \cap H$. We deduce that when $F_3 \neq 1$, $O_\pi(K \cap H)$ is a Hall π-subgroup of $N_H(P)$.

In order to prove the lemma, we may suppose $F_3 = 1$. By (B1), $m(\Omega_1(Z(P))) \geqslant 3$. By 5.31 and Theorem 6.2.4 of [10], we have

(*) Each $\Omega_1(Z(P))$-invariant $3'$-subgroup of H lies in $C_H(\Omega_1(Z(P)))$.

Using 5.16, (i) and (v), we find that H satisfies the hypotheses of 4.8. Using the notation of 4.8, we see that H, P, U, W_X, π satisfy Hypothesis 4.1 and $N(P) = PC(P)U$.

Let $T^\alpha = T \in S_2(U)$. If $C_T(P) \neq 1$ then, by (B3), (B4) and (B5), $N_H(C_T(P)) \leqslant N_H(P)$. By (iv) and (*), H satisfies the hypotheses of 4.2. By (iii), we have a contradiction. So $C_T(P) = 1$.

So $PC_H(P)$ has odd order. By 5.31, $C_H(a)$ has odd order for each $a \in P^\#$. But now H has no elements of order 6. By 5.10, H is solvable, in contradiction with (ii). The lemma is proved.

Proof of 5.14. We claim that H satisfies the hypotheses of 4.3. It follows from 5.16 and 5.32(i) and (v) that H and P satisfy the hypotheses of 4.8. So, using the notation of 4.8, H, P, U, W_X, π satisfy Hypothesis 4.1.

By 5.32(iv), $2 \in \pi$. Also $N_H(P) = PC_H(P)U$. By 5.31, H satisfies hypothesis (C3) of 4.3. But, by 2.2 and 5.32(vi), H, P, U, W_X and π satisfy the hypotheses of 4.3. But now, by 5.32(iii), we have a contradiction. So no such G can exist and 5.14 is proved.

We now prove 5.5 and 5.6. We require a lemma.

LEMMA 5.33. *Assume Hypothesis* 5.1, *Let* $q \in \pi(G)$ *and* $Y = O_q(G)$. *For any subgroup* H *of* G, *let* $\bar{H} = HY/Y$. *Then, if* $G \neq YF$,

(i) $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$.

(ii) $\bar{F} = C_{\bar{G}}(\alpha)$.

(iii) $\bar{K} = \bar{F} \cap N_{\bar{G}}(\bar{P})$.

(iv) $\bar{G}$ *satisfies Hypothesis* 5.1 *with* P *replaced by* $\bar{P}$.

(v) *Let* $r \in \pi(G)$ *and* R *be an* α-*invariant Sylow* r-*subgroup of* G. *Let* $W_R \leqslant Z(R)$ *such that* $N(W_R) = N(R)$ *and* W_R *is weakly closed in* R *with respect to* G. *Then* $\bar{W}_R$ *is weakly closed in* R *with respect to* G *and* $N_{\bar{G}}(\bar{W}_R) = N_{\bar{G}}(\bar{R}) = \overline{N(R)}$.

(vi) *If* G, P, W *satisfies Hypothesis* 5.2 *then* $\bar{G}$, $\bar{P}$, $\bar{W}$ *satisfies Hypothesis* 5.2.

(vii) *Suppose that whenever* H *is an* α-*invariant solvable subgroup of* G *such that* $P \cap H \in S_3(H)$ *and* $Z(K) \leqslant H$ *then* $P \cap H \trianglelefteq H$. *Then whenever* $\bar{H}$ *is an* α-*invariant solvable subgroup of* $\bar{G}$ *such that* $\bar{P} \cap \bar{H} \in S_3(\bar{H})$ *and* $Z(\bar{K}) \leqslant \bar{H}$ *then* $\bar{P} \cap \bar{H} \trianglelefteq \bar{H}$.

Proof. (i) follows as with 4.9(i).

(ii) follows from Theorem 6.2.2 of [10].

Let N be the pre-image in G of $N_{\bar{G}}(\bar{P})$. By (ii) and Theorem 6.2.2 of [10],

$$C_{\bar{G}}(\alpha) \cap N_{\bar{G}}(\bar{P}) = \overline{C_N(\alpha)} = \bar{F} \cap N_{\bar{G}}(\bar{P}).$$

By (i), $N = YN(P)$. So, by 2.3(ii), $C_N(\alpha) = C_Y(\alpha)K$. So $\overline{C_N(\alpha)} = \bar{K}$. We conclude that (iii) holds.

Suppose that $\bar{G} = \bar{F}$. Then $G = YF$ and we have a contradiction. So $\bar{G} \gneqq \bar{F}$. Now $\bar{P}$ is an α-invariant Sylow 3-subgroup of $\bar{G}$. Let $t \in \pi(\bar{F})$ and $\bar{R}^\alpha = \bar{R} \in S_t(\bar{G})$. Let Q^* be the pre-image in G of $\bar{R}$ and $Q^\alpha = Q \in S_t(Q^*)$. By (ii), $t \in \pi(F)$. So, by (A1), $\bar{Q}$ is the only α-invariant Sylow t-subgroup of $\bar{G}$. We conclude that (iv) holds.

Let r, R and W_R be as in (v). Let $g \in G$ such that $\bar{W}_R^{\bar{g}} \leqslant \bar{R}$. Then $W_R^g \leqslant YR$.

There is a $y \in Y$ such that $W_R^{gy} \leqslant R$. But now $W_R^{gy} = W_R$. So $\bar{W}_R{}^{\bar{g}} = \bar{W}_R$ and $\bar{W}_R$ is weakly closed in $\bar{R}$ with respect to $\bar{G}$. By 4.6(i),

$$N_{\bar{G}}(\bar{R}) \leqslant N_{\bar{G}}(W_{\bar{R}}). \tag{38}$$

We claim that $N_{\bar{G}}(\bar{W}_R) = \overline{N(W_R)}$. By 4.21(i), we may suppose that $q = r$. Let L be the pre-image in G of $N_{\bar{G}}(\bar{W}_R)$. Then $YW_R \trianglelefteq L$. As YW_R is an r-group, by 4.6(i), $W_R \trianglelefteq L$. So, again, $N_{\bar{G}}(\bar{W}_R) = \overline{N(W_R)}$.

As $N(W_R) = N(R)$, it follows from (38) that

$$N_{\bar{G}}(\bar{W}_R) = \overline{N(W_R)} = \overline{N(R)} \leqslant N_{\bar{G}}(\bar{R}) \leqslant N_{\bar{G}}(\bar{W}_R).$$

We have (v).

By (iv) and (v), if G, P, W satisfy Hypothesis 5.2, then $\bar{G}$, $\bar{P}$, $\bar{W}$ satisfy Hypothesis 5.1, (A2) and (A3). But, by Theorem 6.2.2(iv) of [10], $C_{\bar{W}}(\alpha) = \overline{C_{YW}(\alpha)}$. By 2.3(ii), $C_{\bar{W}}(\alpha) = \overline{C_W(\alpha)}$. By (A4), we have (A4) for $\bar{G}$, $\bar{P}$, $\bar{W}$. We have (vi).

Assume the hypothesis of (vii), Let $\bar{H}$ be an α-invariant solvable subgroup of $\bar{G}$ such that $\bar{P} \cap \bar{H} \in S_3(\bar{H})$ and $Z(\bar{K}) \leqslant \bar{H}$. Let H be the pre-image in G of $\bar{H}$. Then H is α-invariant and solvable. By (iii), $Z(K) \leqslant H$. As $O_3(G) \leqslant P$, $P \cap H \in S_3(H)$. So, by the hypothesis of (vii), $P \cap H \trianglelefteq H$. So $\bar{P} \cap \bar{H} \trianglelefteq \bar{H}$ and we have (vii). The lemma is proved.

Proof of 5.5. Let G be a minimal counterexample to 5.5. Let $q \in \pi(G)$ and set $Y = O_q(G)$. For $H \leqslant G$, set $\bar{H} = HY/Y$. Clearly $G \neq YF$.

By 5.33(iv), $\bar{G}$, $\bar{P}$ satisfy Hypothesis 5.1. By 5.33(ii), as F is nilpotent, $O_3(C_{\bar{G}}(\alpha)) = 1$. Also, by 5.33(i), $N_{\bar{G}}(\bar{P}) = F(N_{\bar{G}}(\bar{P})) \cdot C_{N_{\bar{G}}(\bar{P})}(\alpha)$. By 5.33(vii), 5.5(a), (b), (c) hold for $\bar{G}$.

Suppose that there is an α-invariant $W \leqslant \Omega_1(Z(P))$ such that W weakly closed in P with respect to G and $N(W) = N(P)$. By 5.33(v), $\bar{W}$ is weakly closed in $\bar{P}$ and $N_{\bar{G}}(\bar{W}) = N_{\bar{G}}(\bar{P})$. So we can assume that π^* exists and 5.5(d), (e) and (f) apply.

Let $r \in \pi^*$ and $R^\alpha = R \in S_r(G)$, so that $\bar{R} \in S_r(\bar{G})$. By 5.33(v), 5.5(e) holds for $\bar{G}$, $\bar{P}$ and $\bar{W}_R$ and

$$N_{\bar{G}}(\bar{W}_R) = N_{\bar{G}}(\bar{R}) = \overline{N(R)} = \bar{R} \cdot \bar{F}.$$

As $W_R \leqslant Z(R) \cap K$, $\bar{W}_R \leqslant \overline{Z(R) \cap K} \leqslant Z(\bar{R}) \cap \bar{K}$. So, by 5.33(iii), $\bar{W}_R \leqslant Z(\bar{R}) \cap C_{N_{\bar{G}}(\bar{P})}(\alpha)$ and 5.5(d) holds for $\bar{G}$, $\bar{P}$, $\bar{W}_R$.

By 5.5(b), $YN(P)$ is solvable. So, by 5.5(c), $Y \leqslant N(P)$ and so

$$Y \leqslant F(N(P)). \tag{39}$$

An easy argument using (39) and 5.5(f) shows that 5.5(f) holds for $\bar{G}$ and $\bar{P}$. We conclude that $\bar{G}$ satisfies the hypotheses of 5.5. By (39), as $P \ntrianglelefteq G$, $\bar{P} \ntrianglelefteq \bar{G}$. So $\bar{G}$ is a counterexample to 5.5. By minimality of G, $Y = 1$. We conclude that

$$F(G) = 1. \tag{40}$$

We now prove

(I) *Let* $H^\alpha = H \lneqq G$ *such that* $P \cap H \in S_3(H)$ *and* $Z(K) \leqslant H$. *Then, if* $N(\Omega_1(Z(P \cap H))) \leqslant N(P)$, $H \leqslant N(P)$.

Let $Q = P \cap H$ and $Z = \Omega_1(Z(Q))$. By Theorem 6.2.2 of [10], as G, P satisfy Hypothesis 5.1, H and Q satisfy Hypothesis 5.1 or $H \leqslant F$. As $N(Z) \leqslant N(P)$, we can suppose $H \nleqslant F$.

By 5.5(a), $O_3(C_H(\alpha)) = 1$. By 5.5(b) and 5.13, $N_H(Q) = QC_H(Q)(K \cap H)$. As $N(Z) \leqslant N(P)$ and $N(Q) \leqslant N(Z)$, $N_H(Q) = N_H(Z) = N(P) \cap H$. Let $X = N_H(Q) \cap F$ and L be an α-invariant solvable subgroup of H such that $Q \cap L \in S_3(L)$ and $Z(X) \leqslant L$. As $Z(K) \leqslant N_H(Q) = N_H(P)$, $Z(K) \leqslant X \leqslant K$, and so $Z(K) \leqslant Z(X) \leqslant L$. Also $P \cap L = Q \cap L \in S_3(L)$. By 5.5(c), $Q \cap L \trianglelefteq L$. So 5.5(a), (b), (c) hold for H.

Suppose that there is a $W \leqslant \Omega_1(Z(P))$ such that W is weakly closed in P with respect to G and $N(W) = N(P)$. Let $h \in H$ such that $Z^h \leqslant Q$. Then Z, $Z^h \leqslant P$. By 4.6(iii), there is an $n \in N(P)$ such that $Z^h = Z^n$. So $h \in N(P) \cap H = N_H(Z)$. So $Z^h = Z$. Thus Z is weakly closed in Q with respect to G.

So we can suppose that π^* exists and 5.5(d), (e) and (f) apply. Let $r \in \pi^*$ and $R^\alpha = R \in S_r(G)$. Set $R_0 = R \cap H$. As G is a counterexample, $W_R \neq 1$ and $r \in \pi(F)$. By (A1) and Theorem 6.2.2 of [10], $R_0 \in S_r(H)$. By 5.5(d), $W_R \leqslant Z(K) \leqslant H$. So $W_R \leqslant R_0 \leqslant R$.

By 5.5(d), (e) and 4.6(i), $N_H(R_0) = N(R) \cap H = N_H(W_R) = RF \cap H$. By Theorem 2.4.1 of [15], $RF \cap H = R_0 C_H(\alpha)$. By 4.6(i), 5.5(d), (e) hold for H. But, as $F(N(P)) \cap N_H(Q) \leqslant F(N_H(Q))$, it quickly follows that 5.5(f) holds for H. So H satisfies the hypotheses of 5.5 and so, as $H \lneqq G$, $Q \trianglelefteq H$. We conclude that $H \leqslant N(P)$. We have (I).

(II) *Let* Q *be a non-identity* $Z(K) \cdot \langle\alpha\rangle$*-invariant subgroup of* P. *Then* $N(Q) \leqslant N(P)$.

Suppose that (II) is false and choose a counterexample Q such that $|N_P(Q)|$ is maximal. By (40), $N(Q) \lneqq G$. Let $R = P \cap N(Q)$ and $S = \Omega_1(Z(R))$.

If $N(S) \leqslant N(P)$ then $N(R) \leqslant N(P)$. So $R \in S_3(N_{N(Q)}(R))$ and so $R \in S_3(N(Q))$. By (I), $N(Q) \leqslant N(P)$, a contradiction. So $N(S) \nleqslant N(P)$. By maximality of Q,

$$|N_P(R)| \leqslant |N_P(S)| \leqslant |N_P(Q)| = |R|.$$

So $R = N_P(R)$ and so $R = P$. Using the argument of (I), we see that $N(Q)$ and P satisfy the hypotheses of 5.5. But, as $N(Q) \lneqq G$, $P \trianglelefteq N(Q)$ and so $N(Q) \leqslant N(P)$, a contradiction. We conclude that (II) holds.

(III) *Let* $H^\alpha = H \lneqq G$ *such that* $P \cap H \in S_3(H)$ *and* $Z(K) \leqslant H$. *Then* H *is solvable and* $P \cap H \trianglelefteq H$.

By (I), (II) and 5.5(b), we may suppose that $P \cap H = 1$. But now $3 \notin \pi(H)$. So, by 5.10, H is solvable. We have (III).

By 5.5(a), (40) and (III), G, P satisfy Hypothesis 5.8. By 5.14, we have a contradiction. So no such G can exist and the theorem is proved.

Proof of 5.6. Let G be a minimal counterexample to 5.6. Set $P^* = [P, \alpha]$. Let $q \in \pi(G)$ and $Y = O_q(G)$. For $H \leqslant G$, set $\bar{H} = HY/Y$. Clearly $G \neq YF$. By 5.33(vi), $\bar{G}$, $\bar{P}$, $\bar{W}$ satisfy Hypothesis 5.2. We conclude from 5.33(iii), that $\bar{G}$ satisfies the hypotheses of 5.6.

Suppose that $Y \neq 1$. Then, by minimality of G, $\bar{P}^* \trianglelefteq \bar{G}$. So $YP^* \trianglelefteq G$. By 2.6, G/YP^* is 3-nilpotent. But now, by 5.10, G is solvable. By Theorem 6.3.3 of [10], $W \leqslant O_{3',3}(G)$. Let $Q \in S_3(O_{3',3}(G))$ such that $Q \geqslant W$. By 4.6(i), (A2) and (A3), $N(Q) \leqslant N(P) \leqslant N(P^*)$. So, by the Frattini argument,

$$G = O_{3',3}(G) \cdot N(Q) = O_{3'}(G) \cdot N(Q) = O_{3'}(G) \cdot N(P^*).$$

As $P^* \ntrianglelefteq G$, $[P^*, O_{3'}(G)] \neq 1$. But, if $q = 3$, $YP^* \leqslant O_3(G)$, a contradiction. So $q \neq 3$. Consider YPF. This is an α-invariant solvable subgroup of G. By 2.14, $P^* \leqslant O_3(YPF)$. So, as $YP^* \trianglelefteq G$, $P^* = O_3(YP^*) \trianglelefteq G$, a contradiction. So $Y = 1$. We conclude that $F(G) = 1$.

We now prove

(I) $N(P^*) = PC(P^*)F$. *Also* $N(P^*)/P^*$ *is* 3-*nilpotent.*

Let $N = N(P^*)$. By 2.6, N/P^* is 3-nilpotent. By 5.10, $O_{3'}(N/P^*)$, and hence N, is solvable. So, by Theorem 6.3.3 of [10], $W \leqslant O_{3',3}(N)$. Let $Q_0 \in S_3(O_{3',3}(N))$, $Q_0 \geqslant W$. By 4.6(ii), (A2) and (A3), $N(Q_0) \leqslant N(P)$. By (A2), $N(P) \leqslant N(P^*)$. So, by the Frattini argument,

$$N = O_{3',3}(N)\, N_N(Q_0) = O_{3'}(N) \cdot N_N(Q_0) = O_{3'}(N) \cdot N(P).$$

But $O_{3'}(N) \leqslant C(P^*)$. By (A2), $N(P) \leqslant PC(P^*)F$. So we have (I).

(II) *Let R be a non-identity α-invariant subgroup of P. Suppose that $C_W(\alpha) \leqslant R$. Then $N(R) \leqslant N(P)$.*

If $W \leqslant F$ then, by (A2), (A3) and 4.6(i), $N(R) \leqslant N(P)$. By (A4), we may suppose that $C_W(\alpha) = 1$. Set $M = N(R)$ and $S = N_P(R)$. Then $W \leqslant S$. By (A2), (A3) and 4.6(i), $N(S) \leqslant N(P)$. So $S \in S_3(N_M(S))$ and so $S \in S_3(M)$. We now verify the hypotheses of 5.6 for M with P replaced by S and W unchanged.

By Theorem 6.2.2 of [10], M and S satisfy Hypothesis 5.1. By 5.13, $N_M(S) = SC_M(S)(K \cap M)$. Clearly $N_M(S) = N(P) \cap M$. So (A2) holds for S. By 4.6(i), (A3) holds for M. So M, S, W satisfy Hypothesis 5.2. But, as $F = K$, $C_M(\alpha) \leqslant N_M(S)$. So M satisfies the hypotheses of 5.6.

But now, as $M \lneqq G$, $[S, \alpha] \trianglelefteq M$. Also $W \leqslant [S, \alpha]$. By 4.6(i), (A2) and (A3), $M \leqslant N([S, \alpha]) \leqslant N(P)$. We have (II).

(III) *Let* $H^\alpha = H \lneqq G$ *such that* $C_W(\alpha) \leqslant H$ *and* $P \cap H \in S_3(H)$. *Then* $[P \cap H, \alpha] \trianglelefteq H$.

Set $R = P \cap H$ and $T^* = \Omega_1(Z(R)) \cap F$. If $T^* \neq 1$, set $T = T^*$. If $T^* = 1$, set $T = \Omega_1(Z(R))$. As $W \leqslant \Omega_1(Z(P))$, $C_W(\alpha) \leqslant T^* \leqslant T$. As $C_W(\alpha) \leqslant R$, by (II), $N_H(R) = N(P) \cap H$. So, by 5.13,

$$N_H(R) = RC_H(R)(K \cap H). \tag{41}$$

By (41), we have that $T \trianglelefteq N_H(R)$. Let $h \in H$ such that $T^h \leqslant R$. Then $T^h \leqslant P$. By 4.6(iii), (A2) and (A3), there is an $n \in N(P)$ such that $T^h = T^n$. So $hn^{-1} \in N(T)$. As $C_W(\alpha) \leqslant T$, by (II), $h \in N(P)$. So $h \in N(P) \cap H = N_H(R)$.

It follows that, as $T \trianglelefteq N_H(R)$, $T^h = T$. So T is weakly closed in R with respect to H. We have shown that if $h \in N_H(T)$ then $T^h \leqslant R$, so that $h \in N_H(R)$. As $T \trianglelefteq N_H(R)$, and $N_H(R) = N(P) \cap H$, we obtain that $N_H(R) = N_H(T) = N(P) \cap H$. Using Theorem 6.2.2 of [10], as $C_T(\alpha) = 1$ or $T \leqslant F$, we see that H, R, T satisfies Hypothesis 5.2.

But, as $K = F$, $C_H(\alpha) \leqslant N(P) \cap H = N_H(R)$. So H satisfies the hypotheses of 5.6. By minimality of G, $[R, \alpha] = [P \cap H, \alpha] \trianglelefteq H$. We have (IV).

(IV) $P = P^*$.

Set $L = O^3(G)$. We argue that $L = G$. Now $P \cap L \in S_3(L)$. Set $L^* = LC_W(\alpha)$. Suppose that $L^* \lneqq G$. Then, by 2.6 and (III), $L^*/[P \cap L^*, \alpha]$ is 3-nilpotent. By 5.10, L^* is solvable. We deduce that G is solvable, a contradiction as $F(G) = 1$. So $L^* = G$.

Suppose that $L \lneqq G$. Then $C_W(\alpha) \neq 1$. By (A4), $W \leqslant F$ and $G = LW$. We claim that $P^* \in S_3(L)$ and $P = P^*W$. Clearly $P^* \leqslant L$. By (I), $N(P^*)/P^*$ is 3-nilpotent. So $P^* \in S_3(O_{3'}(N(P^*) \bmod. P^*))$. By (A2), (A3) and Theorem 14.4.2 of [15], as $O^3(N(P^*)) \leqslant O_{3'}(N(P^*) \bmod. P^*)$,

$$P \cap L = P \cap O^3(N(P^*)) \leqslant P \cap O_{3'}(N(P^*) \bmod. P^*) = P^*.$$

As $P^* \leqslant L$, $P^* = P \cap L \in S_3(L)$. As $G = LW$, $P = (P \cap L)W = P^*W$. Let $U^* = \Omega_1(Z(P^*)) \cap F$. Define U to be U^* if $U^* \neq 1$. If $U^* = 1$, let $U = \Omega_1(Z(P^*))$. By (A2), $U \trianglelefteq N(P)$. It quickly follows that $N(U)$ satisfies the hypotheses of 5.6 with P, W unchanged. As $F(G) = 1$ and $U \neq 1$, $N(U) \lneqq G$. By minimality of G, $N(U) \leqslant N(P^*)$. By (I), $U \trianglelefteq N(P^*)$. We conclude that

$$N(U) = N(P^*) \qquad \text{and} \qquad N_L(U) = N_L(P^*). \tag{42}$$

Let $g \in L$ such that $U^g \leqslant P^*$. By 4.6(iii), (A2) and (A3), there is an $n \in N(P) \leqslant N(P^*)$ such that $U^g = U^n$. So $gn^{-1} \in N(U)$. By (42), $g \in N(P^*)$ and $U^g = U$. So U is weakly closed in P^* with respect to L. Using (I) and the argument of 5.13,

$$N_L(P^*) = P^*C_L(P^*)(K \cap L). \tag{43}$$

By Theorem 6.2.2 of [10], L, P^* satisfy Hypothesis 5.1. By weak closure of U, (42) and (43), as $F \leqslant N(P^*)$, L, P^*, U satisfy the hypotheses of 5.6. So $P^* \trianglelefteq L$ and $P^* \trianglelefteq G$, a contradiction. We conclude that $L = G$.

By Theorem 14.4.2 of [10], as G, P, W satisfy Hypothesis 5.2, $P = P \cap O^3(G) = P \cap O^3(N(P))$. By 2.6, $N(P)/P^*$ is 3-nilpotent. As above, $P \leqslant P^*$ and so $P = P^*$. We have (IV).

By (III) and (IV), as $F(G) = 1$, G satisfies the hypotheses of 5.8. By 5.14, we have a contradiction. So no such G can exist and 5.6 is proved.

Before proving 5.7, we need the following lemma.

LEMMA 5.34. *Assume that G satisfies Hypothesis* 5.2. *Let* $Y = O_3(Z(G))$. *For $H \leqslant G$, let $\bar{H} = HY/Y$. Then the following holds, if $P \ntrianglelefteq G$,*

(i) $N_{\bar{G}}(\bar{P}) = \overline{N(P)}$.

(ii) *$\bar{G}$, $\bar{P}$, $\bar{W}$ satisfies Hypothesis* 5.2.

(iii) *If each W-invariant $3'$-subgroup of G lies in $C(W)$ then each $\bar{W}$-invariant $3'$-subgroup of $\bar{G}$ lies in $C_{\bar{G}}(\bar{W})$.*

Proof. (i) follows as does 4.9(i). If $G = YF$ then $P \trianglelefteq G$, a contradiction. So $G \neq YF$. Now (ii) follows as 5.33(vi). (iii) follows as does 4.9(ii). The lemma is proved.

Proof of 5.7. Let G be a minimal counterexample to 5.7 and $Y = O_3(Z(G))$. For $H \leqslant G$, set $\bar{H} = HY/Y$. By 5.34(ii), $\bar{G}$, $\bar{P}$, $\bar{W}$ satisfy Hypothesis 5.2. By Theorem 6.2.2 of [10], 5.7(a) holds for $\bar{G}$. By 5.34(iii), (b) holds for $\bar{G}$. By (c), (d) and 5.34(i), $\bar{G}$ satisfies the hypotheses of 5.7. But, as $Y \leqslant P$, $\bar{P} \ntrianglelefteq \bar{G}$. So, by minimality of G, $Y = 1$.

Suppose that $O_3(G) \neq 1$. Then $1 \neq O_3(G) \trianglelefteq P$. Let $Z = Z(P) \cap O_3(G)$. Then $Z \neq 1$. Let $C = C(Z)$. By 5.13, $N_C(P) = PC_C(P)(K \cap C)$. It follows that C, P, W satisfy Hypothesis 5.2. It quickly follows that C satisfies the hypotheses of 5.7. As $Y = 1$, $C \lneqq G$. By minimality of G, $P \trianglelefteq C$. So $C(O_3(G)) \leqslant C(Z) \leqslant N(P)$. Thus

$$W \leqslant C_P(O_3(G)) = O_3(C(O_3(G))) \trianglelefteq G.$$

By 4.6(i), (A2) and (A3), $P \trianglelefteq G$, a contradiction. We deduce that $O_3(G) = 1$. We now prove

(I) *Let Q be a non-identity $Z(K) \cdot \langle\alpha\rangle$-invariant subgroup of P. Then $N(Q) \leqslant N(P)$.*

Let $N = N(Q)$ and $R = N_P(Q)$. Then $W \leqslant R$. By 4.6(i), (A2) and (A3),

$$N_N(R) = N(P) \cap N. \tag{44}$$

We deduce from 4.6(i) that (A2), (A3) and (A4) hold for N, R, W. By 5.13, $N_N(R) = RC_N(R) \cdot (K \cap N)$. So N satisfies the hypotheses of 5.7. By minimality of G, $R \trianglelefteq N$. By (44), $N \leqslant N(P)$. We have (I).

By 5.10, $N(P)/P$ is solvable. By the hypothesis of 5.7 and (I), it quickly follows that G satisfies the hypotheses of 4.8.

Adopt the notation of 4.8. By 4.8, $\pi = \{2\}$, $U = K_2$ and Hypothesis 4.1 holds for G, P, π, U and W_X. Now $1 \in \mathscr{X}$ and $W_1 = \Omega_1(Z(P))$. It follows from (b) that each W_1-invariant $3'$-subgroup of G lies in $C(W_1)$. By 2.2, as $U = K_2$, we have that if $V \leqslant U$ then $O^2(N(V)) \leqslant C(V)$. So G, P, π, U and W_X satisfy the hypotheses of 4.4.

By 4.4, $P \trianglelefteq G$, a contradiction. So no such G can exist and 5.7 is proved.

6. Proof of the Main Theorem

Assume that the main theorem of this paper is false and let G denote a minimal counterexample. Then it follows from Theorem 6.2.3 of [10] that G is of order prime to p. Using Theorem 6.2.2 of [10], we see that G is not solvable and G is characteristically simple.

Set $\beta = \alpha^p$ and $F = C_G(\beta)$. Then, by Theorem 10.2.1 of [10], F is a nilpotent p'-group. For any prime q, let $F_q = O_q(F)$.

Also, by Theorem 6.2.2 of [10], there is one and only one α-invariant Sylow q-subgroup for each $q \in \pi(G)$. By 6.1, $3 \in \pi(G)$. Let P denote the unique α-invariant Sylow 3-subgroup of G and $K = C_{N(P)}(\beta)$. For any prime q, let $K_q = O_q(K)$.

Lemma 6.1. $2, 3 \in \pi(G)$ *and* $p \geqslant 5$.

Proof. By 5.10, $2, 3 \in \pi(G)$. As $p \notin \pi(G)$, $p \geqslant 5$. The lemma is proved.

Lemma 6.2. *Let H be an α-invariant proper subgroup of G. Then*

(i) $H = F(H)\, C_H(\beta)$.

(ii) *Let $q \in \pi(H)$ and Q be an α-invariant Sylow q-subgroup of H. Then* $[Q, \beta] = [O_q(H), \beta] \trianglelefteq H$.

Proof. (i) follows from 2.1 and minimality of G.

Let $q \in \pi(H)$ and $Q^\alpha = Q \in S_q(H)$. Then $O_q(H) \leqslant Q$. Also, by (i), $[Q, \beta] \leqslant O_q(H)$. So, by Theorem 5.3.6 of [10],

$$[Q, \beta] = [Q, \beta, \beta] \leqslant [O_q(H), \beta] \leqslant [Q, \beta].$$

By (i), $[O_q(H), \beta] = [Q, \beta] \trianglelefteq H$. We have (ii) and the lemma.

We now apply Theorem 1.1.

LEMMA 6.3. *Let $q \in \pi(G) - \{2\}$ and Q be an α-invariant Sylow q-subgroup of G. Then $N(Q)$ is a maximal α-invariant subgroup of G and $N(Q) = N([Q, \beta])$.*

Proof. By 2.6, $[Q, \beta] \neq 1$. Let C be a non-identity characteristic subgroup of Q. Then, by 6.2(i), $N(C) \leqslant N([Q, \beta])$. By Theorem 1.1,

$$Q = Q \cap G' = \langle Q \cap N(C)' : 1 \neq C \text{ char. } Q \rangle.$$

So, by 6.2(ii), $Q \leqslant F(N([Q, \beta]))$. So $N(Q) = N([Q, \beta])$. But, by 6.2(ii), $N([Q, \beta])$ is a maximal α-invariant subgroup of G. The lemma follows.

We believe that the next lemma has been obtained by several other authors.

LEMMA 6.4. *Let $q \in \pi(F)$ and Q be an α-invariant Sylow q-subgroup of G. Then*

(i) $[Q, \beta] \neq 1$.

(ii) $N(F_q) \leqslant N([Q, \beta])$.

(iii) $F \leqslant N(Q)$.

Proof. (i) is a consequence of 2.6, as $G = O^q(G)$.

Let $N = N([Q, \beta])$. Then, by 6.2(ii), N is a maximal α-invariant subgroup of G. Also, by (i) and 6.2(ii),

$$1 \neq [Q, \beta] = [O_q(N), \beta] \trianglelefteq N. \tag{1}$$

Let $W = Z(Q) \cap O_q(N)$ and $W^* = W \cap F$. Then $W \neq 1$. By Theorem 6.2.2 of [10], $W^* \leqslant Z(F)$. By 6.2(i), $Q \leqslant O_{q',q}(N)$. So, by the Frattini argument, $N = O_{q'}(N) \cdot N_N(Q)$. By 6.2(i), as $W^* \leqslant Z(F)$, and by (1), we have that

$$1 \neq W \trianglelefteq N \qquad \text{and} \qquad W^* \trianglelefteq N. \tag{2}$$

If $W^* \neq 1$ then, by maximality of N, $N = N(W^*)$. By Theorem 6.2.2 of [10], $F_q \leqslant Q$. So

$$O_{q'}(F) \leqslant C(F_q) \leqslant C(W^*) \leqslant N.$$

As F is nilpotent, $F \leqslant N$. But now, by 2.2, $N(F_q) \leqslant N$. So, in order to prove (ii), we may suppose that $W^* = 1$.

Let L be a maximal α-invariant subgroup of G such that $L \geqslant N(F_q)$. Then $W \leqslant L$. By 6.2(ii), as $W^* = 1$, $W \leqslant O_q(L)$. By (2) and maximality of N, $N = N(W)$. Let $L^* = [O_{q'}(L), \beta]$ and $N^* = [O_{q'}(N), \beta]$.

Then $O_{q'}(L) \leqslant C(O_q(L)) \leqslant C(W) \leqslant N$. By 6.2(i) and Theorem 5.3.6 of [10],

$$L^* \leqslant N^*. \tag{3}$$

But, by (2), $1 \neq W \leqslant [O_q(L), \beta]$. So, by maximality of L and 6.2(ii), $L = N((O_q(L), \beta])$. By Theorem 6.2.2 of [10], $O_q(L) \leqslant Q \leqslant N$. So, by 6.2(ii), $[O_q(L), \beta] \leqslant O_q(N)$. It follows that $O_{q'}(N) \leqslant L$. By 6.2(i), $N^* \leqslant L^*$. So, by (3), $L^* = N^*$.

Suppose that $L^* \neq 1$. By 6.2(i), $1 \neq L^* \trianglelefteq L$ and $1 \neq N^* \trianglelefteq N$. So, by maximality of L and N, $L = N(L^*) = N(N^*) = N$. So, in order to prove (ii), we may suppose that $L^* = 1$. By 6.2(i), $L = O_q(L) \cdot F$. As F is nilpotent, L is q-closed. It follows that $O_q(L) \in S_q(G)$. By Theorem 6.2.2 of [10], $Q = O_q(L)$. But now, by 6.2(ii), $L = N$. We have (ii).

By (ii), as F is nilpotent, F normalizes $[Q, \beta]$ and F_q. By Theorems 5.3.5 and 6.2.2 of [10], $Q = [Q, \beta] \cdot F_q$. Thus $F \leqslant N(Q)$. We have (iii) and the lemma.

LEMMA 6.5. *Let $q \in \pi(F) - \{2\}$ and Q be an α-invariant Sylow q-subgroup of G. Let $W_Q = \Omega_1(Z(Q)) \cap F$. Suppose that $W_Q \neq 1$. Then*

(i) $N(W_Q) = N(Q) = C(W_Q)$.

(ii) *W_Q is weakly closed in Q with respect to G.*

Proof. By Theorem 6.2.2 of [10], $W_Q \leqslant Z(F_q) \leqslant Z(F)$. So, by 6.2(i), $W_Q \leqslant Z(N(Q))$. By 6.3, as $W_Q \neq 1$, we have (i).

Also, by 6.3, $W_Q \trianglelefteq N(Q) = N(J(Q))$. So, by 3.2 and (i), as $(q - 1)$ does not divide $| N(W_Q) : C(W_Q)|$, we have (ii). The lemma is proved.

Let W_Q be as in 6.5. Define $W \leqslant Z(P)$ as follows. If $W_P \neq 1$, let $W = W_P$. Otherwise let $W = \Omega_1(Z(P))$. Let $\pi_0 = \pi(N(P)/F(N(P)))$, $\pi = \pi_0 - \{2\}$ and $\pi^* = \{q \in \pi : O_q(N(P)) \neq 1\}$. The set π^* corresponds to the π^* of 5.5 as we shall see in 6.7 and 6.10. If $\pi^* = \emptyset$ then $O_\pi(K)$ is a Hall π-subgroup of $N(P)$. In 6.6, we discuss this condition.

LEMMA 6.6. *Suppose that $O_\pi(K)$ is a Hall π-subgroup of $N(P)$. Then β, G, P, W satisfy Hypothesis* 5.2.

Proof. By 6.1 and 6.4(iii), β, G, P satisfy Hypothesis 5.1. By 6.2(i), $N(P) = PC(P)K$. So, by 6.5, we may suppose that $W_P = 1$. By 6.3, $W \trianglelefteq N(P) = N(J(P))$. Let $x \in N(W) - C(W)$. By 6.3, $N(W) = N(P)$ and so $x \in N(P) - F(N(P))$.

By 6.2(i), $x = yz$ where $y \in F(N(P))$ and $z \in K$. Then $[W, x] = [W, z]$, a β-invariant subgroup of W. As $W_P = 1$ and $[W, x] \neq 1$, by 6.1, $m([W, x]) \geqslant 3$. So G, P, W satisfy Hypothesis 3.1.

We apply 3.3. Now $\pi(\mathrm{Aut}_G(W)) - \{2\} \subseteq \pi$. Let Y be a π-subgroup of $N(P)$. As $N(P)$ is solvable there is an $n \in N(P)$ such that $Y^n \leqslant O_\pi(K)$. So, by 2.2 $2 \notin \pi(\mathrm{Aut}_G(Y^n)) = \pi(\mathrm{Aut}_G(Y))$. We deduce that G, P, W satisfy the hypotheses of 3.3.

By 3.3, as $N(W) = N(P) = F(N(P))K$, the lemma is proved.

LEMMA 6.7. $F_3 = 1$.

Proof. Suppose that $F_3 \neq 1$. Now $\pi^* \subseteq \pi(F)$. Let $q \in \pi^*$ and $Q^\alpha = Q \in S_q(G)$. Then, by 6.4(iii), $F \leqslant N(Q)$. So $F_3 \leqslant N(Q)$.

Suppose that $W_P \neq 1$. By 6.5 and 6.1, β, G, P, W satisfy Hypothesis 5.2. By 6.4(iii), $F \leqslant N(P)$. So, by 5.6, $[P, \beta] \trianglelefteq G$. But $O_3(G) = 1$. So $[P, \beta] = 1$, in contradiction with 6.4(i).

So $W_P = 1$ and $W = \Omega_1(Z(P))$. Let $Q_0 = Q \cap N(P)$. By Theorem 6.2.2 of [10], $Q_0 \in S_q(N(P))$. Let L be a maximal α-invariant subgroup of $G, L \geqslant N(Q_0)$. We observe that $[F_3, Q_0] \geqslant Q \cap P = 1$. So $F_3 \leqslant C(Q_0)$.

Also, as $q \in \pi^*$, by 6.3, $N(P) = N(O_q(N(P)))$. So

$$C(Q_0) \leqslant C(O_q(N(P))) \leqslant N(P). \tag{4}$$

Suppose that $F_3 \in S_3(C(Q_0))$. By 6.3 and 6.4(ii), $N(F_3) \leqslant N(P)$. So, by the Frattini argument and (4),

$$N(Q_0) = C(Q_0) \cdot (N(Q_0) \cap N(F_3)) \leqslant N(P). \tag{5}$$

Now suppose that $F_3 \notin S_3(C(Q_0))$. Let $P_0 = P \cap L$. By 6.4(iii) and Theorem 6.2.2 of [10], $P_0 \in S_3(L)$. Then $[P_0, \beta] \neq 1$. By 6.2(ii), $O_3(L) \neq 1$. By maximality of L, $L = N(O_3(L))$. But now $W \leqslant Z(P) \leqslant L$. As $C_W(\beta) = 1$, $W \leqslant O_3(L) \leqslant P_0 \leqslant P$. So $W \leqslant Z(O_3(L)) \leqslant Z(F(L))$. By 6.3, $N(P) = N(W)$. So $F(L) \leqslant N(P)$. By 6.2(i), 6.3 and 6.4(iii), as $F_3 \neq 1$, $L = N(P)$. So $N(Q_0) \leqslant L \leqslant N(P)$, we have, by (5), that in all cases,

$$N(Q_0) \leqslant N(P). \tag{6}$$

But $Q_0 \in S_q(N(P))$. So $Q_0 \in S_q(N(Q_0))$ and so $Q_0 = Q \in S_q(G)$. By (6), $N(Q) \leqslant N(P)$. But, by 6.3, $N(Q) = N(P)$. So $q \notin \pi$ and hence $q \notin \pi^*$, a contradiction.

We conclude that $\pi^* = \emptyset$. By 6.2(ii), $O_\pi(K)$ is a Hall π-subgroup of $N(P)$. So, by 6.6, β, G, P, W satisfy Hypothesis 5.2. Also, by 6.4(iii) and 6.3, $F = K$. So, by 5.6, $[P, \beta] \trianglelefteq G$. Thus $P \leqslant F$, in contradiction with 6.4(i). So $F_3 = 1$ and the lemma is proved.

LEMMA 6.8. *Let Q be a non-identity α-invariant subgroup of P. Then $N(Q) \leqslant N(P)$.*

Proof. Suppose not and choose Q of maximal order subject to violating 6.8. Then $Q \neq P$ and so $Q \neq N_P(Q)$. We deduce that $N(N_P(Q)) \leqslant N(P)$. So $N_{N(Q)}(N_P(Q)) \leqslant N(P) \cap N(Q)$. We conclude that $N_P(Q) \in S_3(N(Q))$. By 6.2(i) and 6.7, $N_P(Q) \trianglelefteq N(Q)$. So $N(Q) \leqslant N(P)$, a contradiction. The lemma is proved.

COROLLARY 6.9. *Let H be an α-invariant proper subgroup of G such that $H \cap P \neq 1$. Then $H \leqslant N(P)$.*

Proof. Let $Q = P \cap H$. By 6.8, $N_H(Q) \leqslant N(P) \cap H$. Thus $Q \in S_3(N_H(Q))$ and so $Q \in S_3(H)$. By 6.7 and 6.2, $Q \trianglelefteq H$. So, by 6.8, $H \leqslant N(P)$. The corollary is proved.

LEMMA 6.10. *Suppose that $\pi^* \neq \emptyset$. Let $q \in \pi^*$ and Q be an α-invariant Sylow q-subgroup of G. Then*

(i) $W_Q = \Omega_1(Z(Q)) \leqslant Z(K)$.

(ii) $N(W_Q) = N(Q) = QF$.

(iii) *W_Q is weakly closed in Q with respect to G.*

Proof. As $q \in \pi$, $N(Q) \neq N(P)$. So, by 6.3 and 6.9, $P \cap N(Q) = 1$. By 6.4(iii) and Theorem 6.2.2 of [10], we have that,

$$3 \notin \pi(N(Q)). \tag{7}$$

If $Q \leqslant N(P)$ then, by 6.2(ii), $[Q, \beta] \trianglelefteq N(Q)$. So, by 6.3, as $N(P) \neq N(Q)$, we have that

$$Q \nleqslant N(P). \tag{8}$$

By 6.3, $N(P) = N(O_q(N(P)))$. By Theorem 6.2.2 of [10], $O_q(N(P)) \leqslant Q$. Set $A = \Omega_1(Z(Q))$. It follows that $A \leqslant N(P)$.

Suppose that $[A, \beta] \neq 1$. Then, by 6.2(i), $[A, \beta] \leqslant O_q(N(P))$. So, by 6.9, $Q \leqslant C([A, \beta]) \leqslant N(P)$, in contradiction with (8). So $A \leqslant F \cap N(P) = K$. We conclude that $A = W_Q$. As $W_Q \leqslant Z(Q)$,

$$W_Q \leqslant Z(K_q) \leqslant Z(K). \tag{9}$$

We have (i).

By 6.3 and 6.5, it is enough to show that $N(Q) = QF$. By 6.2, we may suppose that there is an $r \in \pi(F(N(Q))) - \{q\}$ such that $[O_r(N(Q)), \beta] \neq 1$. By 6.3 and Theorem 6.2.2 of [10], $O_r(N(Q)) \leqslant C(Q) \leqslant C(O_q(N(P))) \leqslant N(O_q(N(P))) = N(P)$. By 6.2, $[O_r(N(Q)), \beta] \leqslant O_r(N(P))$. So, by (7), $P \leqslant C([O_r(N(Q)), \beta])$. So, by 6.9, $Q \leqslant C([O_r(N(Q)), \beta]) \leqslant N([O_r(N(Q)), \beta]) \leqslant N(P)$, in contradiction with (8). The lemma follows.

LEMMA 6.11. *Let $q \in \pi_0$. Then $C_P(Z(K_q)) = 1$.*

Proof. Suppose that $C_P(Z(K_q)) \neq 1$. Let $X = F_q \cap N(K_q)$. By 6.2, $\pi_0 \subseteq \pi(K)$. So, by 6.9,

$$C(K_q) \leqslant C(Z(K_q)) \leqslant N(P). \tag{10}$$

Let $Q^\alpha = Q \in S_q(G)$. By 6.2, 6.4(i) and Theorem 5.3.4 of [10],

$$1 \neq [C_Q(K_q), \beta] \leqslant [O_q(C(K_q)), \beta].$$

By (10) and 6.2, $[O_q(C(K_q)), \beta] \leqslant O_q(N(P))$. So, by 6.9, $N([O_q(C(K_q)), \beta]) \leqslant N(P)$. But X normalizes K_q and so X normalizes $C(K_q)$. As $X \leqslant F$,

$$X \leqslant N([O_q(C(K_q)), \beta]) \cap F_q \leqslant N(P) \cap F_q = K_q .$$

It follows that $F_q = K_q$. But, by (10), $O_{q'}(F) \leqslant C(K_q) \leqslant N(P)$. So, as F is nilpotent, we have that

$$F \leqslant N(P). \tag{11}$$

Let H be a β-invariant solvable subgroup of G such that $P \cap H \in S_3(H)$ and $Z(K) \leqslant H$. Then, by (11), $C_H(\beta)$ normalizes $P \cap H$. By 6.7 and 2.14, $P \cap H \trianglelefteq H$.

By 6.7, 6.2, 6.6 and 6.10, the definition of π^* and the above observation, it follows that β, G, P satisfy the hypotheses of 5.5. But now, by 5.5, $P \trianglelefteq G$. So $P = 1$, in contradiction with 6.1. So $C_P(Z(K_q)) = 1$ and the lemma is proved.

Lemma 6.12. $|\pi_0| \geqslant 2$.

Proof. Suppose that $|\pi_0| \leqslant 1$. We argue that $\pi_0 = \{2\}$ and apply 5.3, 5.4, 5.5 and 5.7. We consider three cases viz. $2 \notin \pi_0$; $\pi_0 = \{2\}$ and $O_2(N(P)) \neq 1$; $\pi_0 = \{2\}$.

Note that $O_3(G) = 1$ and that, by 6.7, $F_3 = 1$. Also recall that $1 \neq W \leqslant Z(P)$, so that $F(N(P)) \leqslant C(W)$.

Case I. $2 \notin \pi_0$. Now $|N(P) : F(N(P))|$ is odd. By 6.2, $W \trianglelefteq N(J(P)) = F(N(P))K = N(P)$. By 6.3, $N(W) = N(P)$. So, by 3.2, W is weakly closed in P with respect to G. Also $K_2 \leqslant C(P)$. But now, by 5.3, $G = O_{3'}(G)\, N(P)$. As G is characteristically simple, by 6.1, we have a contradiction. So this case is impossible.

Case II. $\pi_0 = \{2\}$ and $O_2(N(P)) \neq 1$. Let $T^\alpha = T \in S_2(G)$. By Theorem 6.2.2 of [10] and 6.3, $N(P) = N(O_2(N(P)))$ and $O_2(N(P)) \leqslant T \cap N(P) \in S_2(N(P))$. We deduce that $Z(T) \leqslant N(P)$. By 6.2, $[Z(T), \beta] \leqslant O_2(N(P))$. If $[Z(T), \beta] \neq 1$ then, by 6.9, $T \leqslant C([Z(T), \beta]) \leqslant N(P)$. So, by 6.11,

$$Z(F_2) \leqslant N(P) \qquad \text{and} \qquad C_P(Z(F_2)) = 1. \tag{12}$$

Now suppose that $Z(T) \leqslant F$. If $C_P(Z(T)) \neq 1$ then, by 6.9, $T \leqslant C(Z(T)) \leqslant N(P)$ and again (12) holds.

Define X to be $Z(F_2)$, if $Z(T) \nleqslant F$ or $C_P(Z(T)) \neq 1$. Otherwise let $X = Z(T)$. By Theorem 6.2.2 of [10], we have proved

(*) $X \leqslant Z(F_2) \cap N(P)$ and $C_P(X) = 1$. Also $X \leqslant Z(K)$.

Let H be a β-invariant solvable subgroup of G such that $P \cap H \in S_3(H)$ and $Z(K) \leqslant H$. By (*) and 2.13, $C_{P\cap H}(X) = 1$ and so $P \cap H \leqslant O_3(H) \trianglelefteq H$. So $P \cap H \trianglelefteq H$.

We conclude that, by 6.1, 6.2, 6.4(iii), 6.6, 6.7, 6.10, the above observation and the definition of π^*, that β, G, P satisfy the hypotheses of 5.5. By 5.5, $P \trianglelefteq G$, in contradiction with 6.1. This eliminates Case II.

Case III. $\pi_0 = \{2\}$ and $O_2(N(P)) = 1$. By 6.2, $K_2 \in S_2(N(P))$. Suppose that $Y = K_2 \cap C(W) \neq 1$. By 6.9, $N(Y) \leqslant N(P)$. Let $T^\alpha = T \in S_2(G)$. Then, by 6.9 and Theorem 6.2.2 of [10], $C_T(K_2) \leqslant C_T(Y) \leqslant T \cap N(P) = K_2$. So $T = K_2$. But this contradicts 6.4(i). So $Y = 1$.

Now, by 6.3, $F(N(P)) \leqslant C(W) \leqslant N(W) = N(P)$. Now, as $K_2 \in S_2(N(P))$, $N(P) = F(N(P))K_2$. So $C(W) = F(N(P))Y = F(N(P))$. We deduce that $C(W) = PC(P)$. By 6.2, 6.3 and 6.7, β, G, P, W satisfy the hypotheses of 5.4.

Adopting the notation of 5.4, we see that if $q \in \pi(G) - \{3\}$, $\mathcal{U}^*_G(W; q)$ is permuted transitively by $C(W)$. Now α permutes the members of $\mathcal{U}^*_G(W; q)$. So, as $p \notin \pi(G)$, there is a $Q \in \mathcal{U}^*_G(W; q)$ such that $Q = Q^\alpha$. By 6.9, $W \trianglelefteq QW$ and so $[Q, W] \leqslant Q \cap W = 1$. We deduce that W centralizes every q-subgroup of G that it normalizes, for each $q \in \pi(G) - \{3\}$.

By Theorem 6.2.2 of [10], W centralizes every $3'$-subgroup of G that it normalizes. Also, as $\pi_0 = \{2\}$ and $K_2 \in S_2(N(P))$, by 6.6 and 6.7, β, G, P, W satisfy the hypotheses of 5.7. So, by 5.7, $P \trianglelefteq G$. But this contradicts 6.1.

The lemma follows.

Proof of the Main Theorem. Let H be a β-invariant solvable subgroup of G such that $P \cap H \in S_3(H)$ and $Z(K) \leqslant H$. By 6.12, there are distinct primes q, $r \in \pi_0$. By 6.11, $C_P(Z(K_q)) = C_P(Z(K_r)) = 1$.

Let $t \in \pi(F(H)) - \{3\}$. We may suppose that $q \neq t$. Let $S = O_t(H) \cdot (P \cap H) \cdot Z(K_q)$. Now $O_t(H) \cdot (P \cap H) \leqslant O_{q'}(S)$. By 2.11,

$$P \cap H = [P \cap H, Z(K_q)] \leqslant [O_{t'}(S), Z(K_q)] \leqslant F(S). \tag{13}$$

Let $L = O_{3'}(H)$. We have proved that

$$[P \cap H, O_{3'}(F(H))] = [P \cap H, F(L)] = 1.$$

By 2.2 and Theorem 6.1.3 of [10],

$$L = F(L) \cdot C_L(P \cap H) \leqslant C_H(P \cap H). \tag{14}$$

By the main theorem of [18], H has Fitting height 3. Set $Y = F_2(H)$. By (13), $P \cap H = [P \cap H, Z(K_q)] \leqslant Y$. By the Frattini argument, as $P \cap H \leqslant O_{3',3}(Y)$,

$$H = O_{3',3}(Y) \cdot N_H(P \cap H) = O_{3'}(Y) \cdot N_H(P \cap H).$$

It follows from (14) that $P \cap H \trianglelefteq H$.

By 6.1, 6.2, 6.4(iii), 6.6, 6.7, 6.10, the definition of π^* and the above observation, β, G, P satisfy the hypotheses of 5.5. By 5.5, $P \trianglelefteq G$, in contradiction with 6.1.

We conclude that no minimal counterexample to the main theorem can exist and so the main theorem is proved.

Acknowledgments

This work was done during the tenure of an L. E. Dickson Instructorship at the University of Chicago and the tenure of a Junior Research Fellowship of Merton College, Oxford, England. We should like to thank these bodies and the National Science Foundation for their kind support. The author is now with GEC, Ltd.

Finally we should like to thank the referee of this paper for his work and his valuable suggestions.

References

1. J. Carr, On groups admitting a fixed-point-free automorphism of order the square of a prime, *J. London Math. Soc.* (2) **11** (1975), 129–133.
2. M. J. Collins, D. Phil. thesis, Oxford, 1969.
3. M. J. Collins and B. Rickman, Finite groups admitting an automorphism with two fixed points, to appear.
4. L. Dornhoff, "Group Representation Theory," Part A, Dekker, New York, 1971.
5. W. Feit and J. G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.
6. L. Fletcher and W. B. Stewart, unpublished work on finite simple groups which have no elements of order 6 but have a subgroup isomorphic to A_4.
7. G. Glauberman, "Factorisations in Local Subgroups of Finite Groups," CBMS Series, Amer. Math. Soc., Providence, R.I., in press.
8. G. Glauberman, Weakly closed elements of Sylow subgroups, *Math. Z.* **107** (1968), 1–20.
9. G. Glauberman, Prime power factor groups of finite groups, II, *Math. Z.* **117** (1970), 46–56.
10. D. Gorenstein, "Finite Groups," Harper & Row, New York, 1968.
11. D. Gorenstein and K. Harada, Finite groups whose 2-subgroups are generated by at most four elements, *Mem. Amer. Math. Soc.* No. 147 (1974).
12. D. Gorenstein and I. N. Herstein, Finite groups admitting a fixed point free automorphism of order 4, *Amer. J. Math.* **83** (1961), 71–78.
13. D. Goldschmidt, Solvable signaliser functors on finite groups, *J. Algebra* **21** (1972), 137–148.
14. D. Goldschmidt, A group theoretic proof of the $p^\alpha q^\beta$ theorem for odd primes, *Math. Z.* **113** (1970), 373–375.
15. M. Hall, Jr., "The Theory of Groups," Macmillan Co., New York, 1959.
16. G. Higman, Odd characterisations of finite simple groups, lecture notes, University of Michigan, 1968.
17. G. Higman, Some p-local conditions for odd p, *Inst. Naz. Alta Math. Symposia Math.* **13** (1974).

18. H. Kurzweil, Auflösbare gruppen, die eine abelsche automorphismengruppe gestatten, deren fixpuntgruppen nilpotent ist, *J. Algebra* **10** (1968), 92–101.
19. P. McBride, Near solvable signaliser functors on finite groups, to appear.
20. P. McBride, unpublished notes on known finite simple groups with a strongly 3-embedded subgroup, 1976.
21. M. Pettet, A note on finite groups having a fixed-point-free automorphism, *Proc. Amer. Math. Soc.* **52** (1975), 79–80.
22. M. B. Powell and G. Higman, "Finite Simple Groups," Academic Press, London/New York, 1971.
23. B. Rickman, D. Phil. thesis, Oxford, 1974.
24. E. Shult, On groups admitting fixed-point-free operator groups, *Illinois J. Math.* **9** (1965), 701–720.
25. B. Stellmacher, L. Fletcher, and W. B. Stewart, Endliche Gruppen die kein Element der Ordnug 6 enhalten, to appear.
26. G. Glauberman, in preparation.